§5.4 Degeneracy loci and double Schubert polynomials

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Fulton's Degeneracy Loci Formula

Deneracy Loci formula [Ful91]

Let $w \in \mathcal{S}_n$ be a permutation. If every component of the degeneracy loci $\Omega_w(f)$ has the expected codimension $\ell(w)$, then

$$[\Omega_w(f)] = \mathfrak{S}_w(\overline{\mathbf{a}} \mid \overline{\mathbf{b}}) \in CH^{\ell(w)}(X).$$

Vector Bundles

Classifying Spaces and Chern Classes

Degeneracy Loci and Double Schubert Polynomials

Vector Bundles

Fix a base scheme X. A **vector bundle** over X is a surjective morphism $\pi : E \to X$ which has a *local trivialization*: $\{U_i\} \to X$ such that

$$
\begin{tikzcd}
U_i \times \mathbb{A}^r \arrow[r, phantom, "\cong"] & U_i \times_X E \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & E \arrow[d] \\
& U_i \arrow[r] & X
\end{tikzcd}
$$

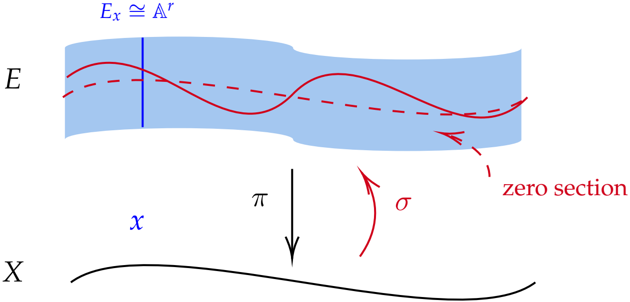
r is called the *rank* of E

For AG students: equivalently, let $\mathcal{E}$ be a locally free sheave of finite rank r. Then E is the *total space*:

$$E = \mathrm{Tot}(\mathcal{E}) := \underline{\mathrm{Spec}}_X(\mathrm{Sym}(\mathcal{E})).$$

Vector Bundles

The *fiber* E_x over $x \in X$ is the pullback of E along $x \to X$, isomorphic to $\mathbb{A}^r$. A *(global) section* to $\pi : E \to X$ is a morphism $\sigma : X \to E$ such that $\pi \circ \sigma = \mathrm{id}_X$. (= an element of the vector space $H^0(X, \mathcal{E})$.) The *zero section* sends every $x \in X$ to $0 \in E_x$.



Slogan

Sections to a vector bundle $E \to X$ are *generalized functions* on X.

Vector Bundles

Usual functorial constructions on vector spaces naturally upgrade to vector bundles:

$$\oplus, \otimes, \ker, \mathrm{coker}, \mathrm{Hom}, (-)^\vee, \mathrm{Sym}, \bigwedge, \det \ldots$$

The *vanishing locus* of a section σ consists of $x \in X$ such that $\sigma(x) = 0$.

Examples:

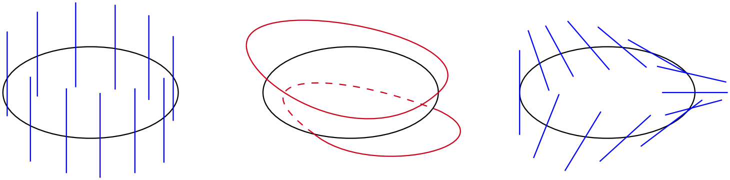
Line bundles over $\mathbb{P}^n$

The algebraic vector bundles over $\mathbb{P}^n$ (any field) are $\mathcal{O}_{\mathbb{P}^n}(d), d \in \mathbb{Z}$. The algebraic sections are

$$H^0(\mathbb{P}^n, \mathcal{O}(d)) = \underbrace{\text{degree } d \text{ polynomials in } x_0, x_1, \ldots, x_n}_{\text{not regular functions, but sections to a line bundle!}}.$$

Vector Bundles

The picture over $\mathbb{R}$:



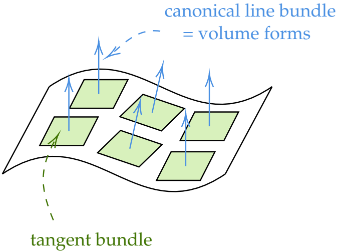
	$\mathcal{O}(0) = \mathcal{O}_{\mathbb{P}^1}$	$\mathcal{O}(1)$	$\mathcal{O}(-1)$ (tautological)
	$H^0(\mathcal{O}_{\mathbb{P}^1}) \cong k$	$H^0(\mathcal{O}(1)) \cong k \cdot \{x, y\}$	$H^0(\mathcal{O}(-1)) = 0$
vanishing locus of generic section	$\mathbb{P}^1$	1 point	No non-zero section

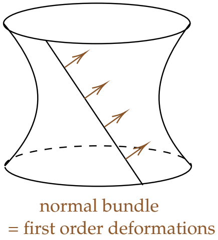
Vector Bundles

Tangent, cotangent, canonical, normal bundles

Let X be a scheme. The *cotangent sheaf* Ω_X is the sheaf of differentials over X. The *tangent sheaf* is the dual $\mathcal{T}_X := (\Omega_X)^\vee$. The *canonical sheaf* is the determinant $\omega_X := \det \Omega_X$. For a closed embedding $X \to Y$, the *normal sheaf* $\mathcal{N}_{X/Y} := (\mathcal{I}_{X/Y}/\mathcal{I}^2_{X/Y})^\vee$ is the difference of tangent sheaves:

$$0 \to \mathcal{T}_X \to \mathcal{T}_Y \to \mathcal{N}_{X/Y} \to 0.$$





Vector Bundles

Tautological bundles

For spaces like $\mathbb{P}^n$, $Gr(k,n)$, Fl_n which classify linear subspaces, there are *tautological bundles* (*universal subbundles* of the trivial bundle) S where the fiber over $[P]$ is P itself.

The *universal quotient bundles* Q are complements of universal subbundles:

$$0 \to \mathcal{S} \to \mathcal{O}^{\oplus N} \to \mathcal{Q} \to 0.$$

Cheerful fact

$$\mathcal{T}_{Gr(k,n)} \cong \mathrm{Hom}(\mathcal{S}, \mathcal{Q}).$$

Vector Bundles

Classifying Spaces and Chern Classes

Degeneracy Loci and Double Schubert Polynomials

Universal Properties of Grassmannians

$\mathbb{P}^n, Gr(k,n), Fl_n$ *classifies* vector bundles over arbitrary X:

- $X \to \mathbb{P}^n \quad \overset{\sim}{\longleftrightarrow} \quad$ line bundles with $n+1$ sections.
- $X \to Gr(k,n) \quad \overset{\sim}{\longleftrightarrow} \quad$ rank k subbundles of $\mathcal{O}_X^{\oplus n}$.
- $X \to Fl_n \quad \overset{\sim}{\longleftrightarrow} \quad$ completely flagged rank n bundles.

Every VB over X is a *pullback* of the universal subbundle, along a *classifying map* φ:

$$\begin{tikzcd} E \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & S \arrow[d] \\ X \arrow[r, "\varphi"'] & Gr(k,n) \end{tikzcd}$$

Classifying Spaces and Universal Bundles

What if we want to classify all rank n bundles? For this we use the convenient gadget of formal colimits of spaces.

A **classifying space** for rank n vector bundles, denoted BGL_n (GL_n is the structure group for rank n vector bundles), satisfies

$$X \to BGL_n \quad \overset{\sim}{\longleftrightarrow} \quad \text{rank } n \text{ VBs over } X.$$

$$\text{classifying maps} \quad \overset{\sim}{\longleftrightarrow} \quad \text{pullbacks of universal bundle.}$$

Examples of classfying spaces

- $B\mathbb{G}_m = \mathbb{P}^\infty = \lim_{k\to\infty} \mathbb{P}^k$.
- $BGL_n = Gr_n = \lim_{k\to\infty} Gr(k,n)$.
- $BGL = Gr = \lim_{n\to\infty} BGL_n$.

Chow Ring of Classifying Spaces

- $CH^\bullet(\mathbb{P}^\infty) \cong \mathbb{Z}[c_1]$, c_1: *universal first Chern class.*
- $CH^\bullet(Gr_n) \cong \mathbb{Z}[c_1, c_2, \ldots, c_n]$, c_i: *universal i^{th} Chern class.*
- $CH^\bullet(Gr) \cong CH^\bullet(Fl_\infty) \cong \mathbb{Z}[c_1, c_2, \ldots]$. cf. [AF23b]
- Other types: Cohomology of $BSO_{2n}, BSO_{2n+1}, BSp_{2n}, \ldots$

Slogan

c_i represents the *stable limit of Schubert varieties.*

Chern Classes

Recall that

$$\text{VB } E \to X \quad \xleftrightarrow{\sim} \quad \text{classifying map } X \xrightarrow{\varphi} BGL_n.$$

The i^{th} **Chern class** $c_i(E) \in CH^\bullet(X)$ is the pullback $\varphi^*(c_i)$ from

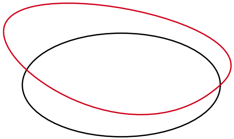
the infinite Grassmannian.

The Chern classes are *shadows of* E: they measure the **degeneracy locus** where i generic sections become linearly dependent. Roughly: how twisted E is.

The top Chern class $c_n(E)$ represents the *vanishing locus of a generic section.*

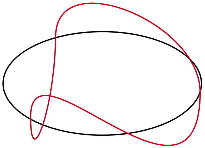
CHERN CLASSES

Some examples of top Chern class:



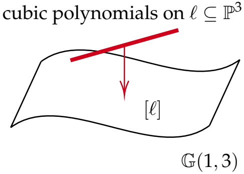
$\mathcal{O}(1)$

one point



$\mathcal{O}(d)$

d points



$\mathrm{Sym}^3\, \mathcal{S}^\vee$

27 lines on a cubic surface

Chern Classes

Chern polynomial $c(E) := 1 + c_1(E) + \cdots + c_n(E)$.

Properties of Chern classes:

- **Whitney sum**

$$0 \to E \to F \to G \to 0 \implies c(F) = c(E)c(G).$$

- **Additive group law** for c_1 of tensor products

$$c_1(E \otimes F) = c_1(E) + c_1(F).$$

- **Functoriality**: commute with pullbacks.

Chern Classes

Splitting principle: can always pretend E splits as $\bigoplus L_i$ (via pulling back to $Fl(E), L_i = E_i/E_{i-1}$).

$$c(E) = \prod(1 + \underbrace{c_1(L_i)}_{\textbf{Chern roots}}).$$

Chern classes are elementary symmetric polynomials in the

Chern roots α_i:

$$c_r(E) = e_r(\alpha_1, \alpha_2, \ldots, \alpha_n).$$

Application: 27 Lines on a Cubic Surface I

$\mathcal{E} := \mathrm{Sym}^3 \mathcal{S}^\vee$ over $\mathbb{G}(1,3) = Gr(2,4)$, $\mathrm{rank}\, \mathcal{E} = 4$.

$$\deg c_4(\mathcal{E}) = \text{ \# lines on a general cubic surface.}$$

Take Chern roots $c(\mathcal{S}^\vee) = (1+\alpha)(1+\beta)$. We have

$$c(\mathcal{S}^\vee) = 1 + s_{\square} + s_{\begin{smallmatrix}\square\\\square\end{smallmatrix}},$$

so

$$\alpha + \beta = s_{\square}, \quad \alpha\beta = s_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}.$$

By splitting principle,

$$\begin{aligned} c(\mathcal{E}) &= c(\mathrm{Sym}^3 \mathcal{S}^\vee) \\ &= (1+3\alpha)(1+2\alpha+\beta)(1+\alpha+2\beta)(1+3\beta) \end{aligned}$$

Application: 27 Lines on a Cubic Surface II

$$
\begin{aligned}
&= (1+3(\alpha+\beta)+9\alpha\beta)\left(1+3(\alpha+\beta)+2(\alpha+\beta)^2+\alpha\beta\right)\\
&= \left(1+3\sigma_{\ydiagram{1}}+9\sigma_{\ydiagram{1,1}}\right)\left(1+3\sigma_{\ydiagram{1}}+2\sigma_{\ydiagram{1}}^2+\sigma_{\ydiagram{1,1}}\right)\\
&= \left(1+3\sigma_{\ydiagram{1}}+9\sigma_{\ydiagram{1,1}}\right)\left(1+3\sigma_{\ydiagram{1}}+3\sigma_{\ydiagram{1,1}}\right)
\end{aligned}
$$

So

$$c_4(\mathcal{E}) = 9\sigma_{\ydiagram{1,1}} \cdot 3\sigma_{\ydiagram{1,1}} = 27\sigma_{\ydiagram{2,2}},$$

and $\deg c_4(\mathcal{E}) = 27$.

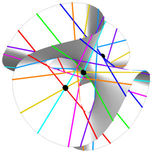


Image from [AMS16]

Vector Bundles

Classifying Spaces and Chern Classes

Degeneracy Loci and Double Schubert Polynomials

Degeneracy Loci

We *relativize* the r^{th} Chern class construction.

The r^{th} **degeneracy locus** of a morphism $f : E \to F$ of vector bundles over X is

$$\Omega_r(f) = \{x \in X \mid \operatorname{rank} f_x \leq r\}\,.$$

If $r = 0$ then we recover the zero locus.

Chern classes as degeneracy loci

Consider the map of bundles over X, $f : \mathcal{O}^{\oplus r} \to E$, defined by generic sections $\sigma_1, \ldots, \sigma_r$. By definition,

$$\Omega_r(f) = \{x \in X \mid (\sigma_1(x), \ldots, \sigma_r(x)) \text{ not full rank }\},$$
$$\implies \quad [\Omega_r(f)] = c_r(E).$$

Degeneracy Loci: Porteous Formula

Relative version of r^{th} Chern class. For any sequence of ring elements $a = (a_1, a_2, \dots)$, let

$$\Delta_f^e(a) := \det \begin{pmatrix} a_f & a_{f+1} & \cdots & \cdots & a_{e+f-1} \\ a_{f-1} & a_f & \cdots & \cdots & a_{e+f-2} \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ a_{f-e+1} & a_{f-e+2} & \cdots & \cdots & a_f \end{pmatrix}$$

Thom-Porteous formula [EH16, Theorem 12.4]

For a bundle map $f : E \to F$ over smooth base, if every component of $\Omega_r(f)$ has the expected codimension, then

$$[\Omega_r(f)] = \Delta_{f-k}^{e-k}\big(c(F-E)\big) := \Delta_{f-k}^{e-k}\left[\frac{c(F)}{c(E)}\right].$$

Degrees of Determinantal Varieties

Application of Porteous Formula [EH16, Theorem 12.5]

A: $e \times f$ matrix of linear forms on $\mathbb{P}^r$. $M_k(A) \subset \mathbb{P}^r$: scheme cut out by $(k+1) \times (k+1)$ minors. If M_k has the expected codimension $(e-k)(f-k)$ in $\mathbb{P}^r$, then

$$\deg(M_k) = \prod_{i=0}^{e-k-1} \frac{i!(f+i)!}{(k+i)!(f-k+i)!}.$$

This includes Segre embeddings $\mathbb{P}^r \times \mathbb{P}^s$, Grassmannians, Flag varieties, etc.

Flagged Degeneracy Loci

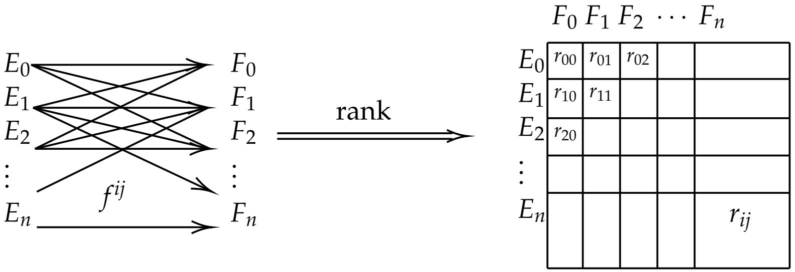
Let $E_\bullet, F_\bullet$ be completely flagged vector bundles with a bundle map

$$0 = E_0 \hookrightarrow E_1 \hookrightarrow \cdots \hookrightarrow E_n = E \xrightarrow{f} F = F_n \twoheadrightarrow \cdots \twoheadrightarrow F_1 \twoheadrightarrow F_0 = 0.$$

Given $w \in \mathcal{S}_n$, define the **flagged degeneracy locus**

$$\Omega_w(f) = \left\{ x \in X \mid \operatorname{rank}\left(f_x^{ij} : (E_i)_x \to \left(F_j\right)_x \right) \leq \operatorname{rk}(w)[i,j] \right\}.$$

Flagged Degeneracy Loci



Slogan

It suffices to check the rank conditions at $\mathrm{Ess}(w)$.

Double Schubert Polynomials

Recall the definition of **double Schubert polynomials**:

$$\begin{aligned}\mathfrak{S}_w(\overline{\mathbf{x}} \mid \overline{\mathbf{y}}) &= \mathfrak{S}_w\left(x_1, \ldots, x_n \mid y_1, \ldots, y_n\right) \\ &= \begin{cases} \prod_{i+j\leq n}\left(x_i - y_j\right) & \text{if } w = [n, n-1, \ldots, 1] \\ \partial_i \mathfrak{S}_{ws_i}(X;Y) & \text{if } w(i) < w(i+1) \end{cases}\end{aligned}$$

Equivalently,

$$\mathfrak{S}_w(\overline{\mathbf{x}} \mid \overline{\mathbf{y}}) = \sum_{\substack{v^{-1}u=w \\ \ell(u)+\ell(v)=\ell(w)}} (-1)^{\ell(v)} \mathfrak{S}_u(\overline{\mathbf{x}}) \mathfrak{S}_v(\overline{\mathbf{y}}).$$

FULTON'S DEGENERACY LOCI FORMULA

Let $f : E_\bullet \to F_\bullet$ be a map of flagged bundles over smooth base X. Let $a_i := c_1(\ker F_i \twoheadrightarrow F_{i-1})$ and $b_j := c_1(E_i/E_{i-1})$.

Deneracy Loci formula [Ful91]

If every component of $\Omega_w(f)$ has the expected codimension $\ell(w)$, then

$$[\Omega_w(f)] = \mathfrak{S}_w(\overline{\mathbf{a}} \mid \overline{\mathbf{b}}) \in CH^{\ell(w)}(X).$$

Proof Idea I

- First consider degeneracy loci D_w of map of flagged *vector spaces*: $V_\bullet \xrightarrow{f} V_\bullet$. Recall: D_w are matrix Schubert varieties.
- $f \in \mathrm{Hom}(V, V)$ has $B = B(V) \times B(V^\vee)$ biaction, under which D_w is invariant. Flags are B-invariant!
- $CH_B(pt) \cong CH_T(pt) \cong \mathbb{Z}[x_1, \ldots, x_n \mid y_1, \ldots, y_n]$ and
$$[D_w]^T = \mathfrak{S}_w(\overline{\mathbf{x}} \mid \overline{\mathbf{y}}).$$
- Relativize to maps of flagged *bundles*. [AF23a] $\square$

PROOF IDEA II

Let $Fl := Fl_\infty$ be the infinite flag variety. Let $\mathcal{S}_\infty := \lim_{n\to\infty} \mathcal{S}_n$.

- Schubert varieties Σ_w indexed by $w \in \mathcal{S}_\infty$ are *universal degeneracy loci*.
- $CH^\bullet_T(Fl) \cong \mathbb{Z}[t_1, t_2, \cdots \mid c_1, c_2, \ldots]$.
- $[\Sigma_w]^T = \mathfrak{S}_w(\bar{\mathbf{t}} \mid \bar{\mathbf{c}})$
- Pullback of $[\Sigma_w]^T$ gives

$$[\Omega_w] = \mathfrak{S}_w(\bar{\mathbf{x}} \mid \bar{\mathbf{y}})$$

where $x_i = c_1(F_i/F_{i-1})$ and $y_i = c_1(\ker E_i \to E_{i-1})$. [AF23b]

$\square$

EXAMPLE 5.49: SCHUBERT VARIETIES

Slogan

Schubert varieties are degeneracy loci!

- Over Fl_n, let $G_\bullet$ be a trivial flagged bundle of rank n. Let $T_\bullet$ be flagged tautological bundle.
- Let $E_\bullet := (\mathcal{O}^{\oplus n}/G_{n-\bullet})^\vee$ and $F_\bullet := T_\bullet^\vee$. Let $f : E_n \to F_n$ be the identity.
- $f_{ij} : E_i \to F_j$ takes a linear functional vanishing on G_{n-i} to its restriction on T_j, so $\operatorname{im} f_{ij} = (T_j/G_{n-i} \cap T_j)^\vee$.
- $\Omega_w(f)$ contains flags $H_\bullet$:

$$\dim \left(H_j/\left(G_{n-i} \cap H_j\right)\right)^\vee = j - \dim \left(G_{n-i} \cap H_j\right) \leq \mathrm{rk}(w)[i,j],$$

  same as $\dim \left(G_i \cap H_j\right) \geq j - \mathrm{rk}(w)[n-i,j]$.
- Conclude that this is the Schubert variety $X_{w_0 w}(G_\bullet)$ (X_w if grade by codimension).

Example 5.49: Schubert Varieties

Set $CH^\bullet(Fl_n) \cong \mathbb{Z}[x_1, \dots, x_n]/(e_1, \dots, e_n)$.
Let's evaluate:

- $\ker F_{i-1} \to F_i$ is the ith dual tautological line bundle $\mathcal{S}_i^\vee$, so $c_1(\ker F_i \to F_{i-1}) = x_i$.
- $E_i/E_{i-1} = \mathcal{O}_{Fl_n}$ trivial so $c_1(E_i/E_{i-1}) = 0$.
- Plugging in: $[\Sigma_{w_0 w}] = \mathfrak{S}_w(\overline{\mathbf{x}} \mid 0) = \mathfrak{S}_w(\overline{\mathbf{x}})$.

Exercise 5.50

Dual map $f^\vee : T_\bullet \to \mathcal{O}^n/G_{n-\bullet}$. Rank conditions $\implies$ $\Omega_{w^{-1}}(f^\vee) = X_{w_0 w}(G_\bullet)$. $c_1(\ker E_i^\vee \to E_{i-1}^\vee) = 0$, and $c_1(F_i^\vee/F_{i-1}^\vee) = c_1(\mathcal{S}_i) = -c_1(\mathcal{S}_i^\vee) = -x_i$. Therefore,

$$\mathfrak{S}_{w^{-1}}(0 \mid -\overline{\mathbf{x}}) = \mathfrak{S}_w(\overline{\mathbf{x}} \mid 0) = \mathfrak{S}_w(\overline{\mathbf{x}}).$$

SYMMETRIES OF SCHUBERT POLYNOMIALS

– Involution.

$$\mathfrak{S}_{w^{-1}}(\overline{\mathbf{x}} \mid \overline{\mathbf{y}}) = \mathfrak{S}_w(-\overline{\mathbf{y}} \mid -\overline{\mathbf{x}}).$$

Proof: dualize $E_\bullet \xrightarrow{f} F_\bullet$.

– **Cauchy formula.**

$$\begin{aligned}\mathfrak{S}_w(x \mid y) &= \sum_{vu \doteq w} \mathfrak{S}_u(x \mid t)\mathfrak{S}_v(t \mid y) \\ &= \sum_{v^{-1}u \doteq w} \mathfrak{S}_u(x \mid t)\mathfrak{S}_v(-y \mid -t).\end{aligned}$$

Proof: decomposition of the diagonal in $Fl \times Fl$.

Example 5.47: Orthogonal Flags I

Fix a (non-Hermitian) inner product on $\mathbb{C}^4$, and consider the locus $Z = \{F_\bullet \in Fl_4(\mathbb{C}) \mid F_3 = F_1^\perp\}$. We calculate the class $[Z]$:

- The inner product is equivalent to an isomorphism $\alpha : \mathbb{C}^4 \to (\mathbb{C}^\vee)^4$ and upgrades to a bundle map

$$T_\bullet : \mathbb{C}^4 \times Fl_4 \to (\mathbb{C}^\vee)^4 \times Fl_4 : T_\bullet^\vee$$

  , flagged with tautological bundles.

- Z is the locus $F_\bullet : \operatorname{rank} \alpha_{31} = 0$, which is

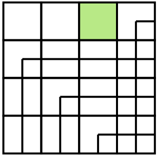
$$Z = \bigcup_{\operatorname{rk} w[3,1]=0} \Omega_w(\alpha)$$

  i.e. $w_1 = 4$. In fact $Z = \Omega_{4123}(w)$, 4123 Bruhat-minimal.

- $\operatorname{Ess}(4123) = \{(3,1)\}$ and $\operatorname{rk}_{4123}[3,1] = 0$.

Example 5.47: Orthogonal Flags II

0	1	1	1
0	1	2	2
0	1	2	3
1	2	3	4



- $c_1\left((T_i/T_{i-1})^\vee\right) = x_i$, and
  $c_1(\ker T_i \to T_{i-1}) = c_1\left((T_i/T_{i-1})^\vee\right) = -x_i$.

- The class of Z is

$$\mathfrak{S}_{4123}(\overline{\mathbf{x}} \mid -\overline{\mathbf{x}}) = 2x_1(x_1+x_2)(x_1+x_3).$$

$\square$

Example 5.48: Isotropic Flags

A flag in Fl_n is *isotropic* if $F_i = F_{n-i}^{\perp}$ for all i. $IFl_n \subseteq Fl_n$ denotes the **isotropic flag variety**. A similar argument shows that

$$IFl_4(\mathbb{C}) = \Omega_{4312}(\alpha), \quad \ell(4312) = 5.$$

However, $\operatorname{codim} IFl_4(\mathbb{C}) = 4$, so $[IFl_4(\mathbb{C})] \neq \mathfrak{S}_{4312}$!

Alert

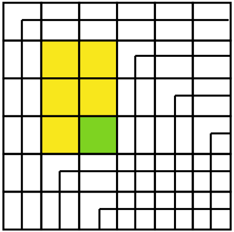
Expected codimension matters!

THEOREM 5.56: PORTEOUS FORMULA I

Let $w(p,q,r)$ be the permutation having diagram $[r+1,q] \times [r+1,p]$:

$$w(p,q,r) = [1 \text{ through } r \mid p+1 \text{ through } p+q-r \mid \text{ rest }]$$

$$w(3,4,1) = [145623]$$
$$\operatorname{Ess} w = \{(3,4)\}$$



Say we have $f : E \to F$ of ranks p, q. We have

$$\Omega_r(f) = \Omega_{w(p,q,r)}(f) \implies [\Omega_r(f)] = \mathfrak{S}_{w(p,q,r)}(\text{ first Chern classes }).$$

Theorem 5.56: Porteous Formula II

Now,

$$\mathfrak{S}_{w(p,q,r)}(x \mid y) = \det \left[e_{q-r+j-i}(x \mid y)\right]_{i,j\in[p-r]}$$

where $e_d(x \mid y) = \sum_{i+j=d}(-1)^j e_i(x)h_j(y)$ is the *double elementary symmetric polynomial*.

Plugging in first Chern classes, we recover

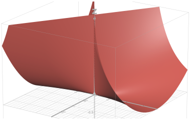
Thom-Porteous Formula [BGP25, Theorem 5.56]

If every component of $\Omega_r(f)$ has codimension $(p-r)(q-r) = \ell(w(p,q,r))$, then

$$[\Omega_r(f)] = \det \left[c_{q-r+j-i}(F-E)\right]_{i,j\in[p-r]}.$$

One More Application: Pinch Points of Surfaces

The projection $\pi : S \to \mathbb{P}^3$ of a smooth surface $S \subset \mathbb{P}^n$ has pinch points where $d\pi : \mathcal{T}_S \to \pi^*\mathcal{T}_{\mathbb{P}^3}$ fails to be injective.



Proposition 12.6 [EH16]

The number of pinch points of a general projection of a smooth surface $S \subset \mathbb{P}^n$ to $\mathbb{P}^3$ is

$$6\deg(S) + \deg\left(4\zeta c_1 + c_1^2 - c_2\right)$$

where the $c_i = c_i\left(\Omega_S\right)$ are the Chern classes of the cotangent bundle of S, and ζ is the hyperplane class.

Optional Extra: Double Grothendieck Polynomials

K-theoretic degeneracy loci formula [[FL94, Theorem 3]]

In $K_0(X)$, let $x_i = 1 - [\ker F_i \to F_{i-1}]$ and $y_j = 1 - [(E_j/E_{j-1})^\vee]$. Then, the class of $\Omega_w(f)$ is represented by a *double Grothendieck polynomial*:

$$[\Omega_w(f)] = \mathfrak{G}_w(\overline{\mathbf{x}} \mid \overline{\mathbf{y}}).$$

References I

[AF23a] David Anderson and William Fulton. “Degeneracy Loci”. In: *Equivariant Cohomology in Algebraic Geometry*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2023, pp. 182–207.

[AF23b] David Anderson and William Fulton. “Infinite-Dimensional Flag Varieties”. In: *Equivariant Cohomology in Algebraic Geometry*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2023, pp. 208–225.

[AMS16] AMS Visual Insight. *27 Lines on a Cubic Surface*. AMS Blog. Feb. 15, 2016. URL: https://blogs.ams.org/visualinsight/2016/02/15/27-lines-on-a-cubic-surface/ (visited on 02/16/2025).

References II

[BGP25] Sara C. Billey, Yibo Gao, and Brendan Pawlowski. "Introduction to the Cohomology of the Flag Variety". In: *Handbook of Combinatorial Algebraic Geometry*. Ed. by Martha Precup et.al. In preparation. 2025.

[EH16] Eisenbud and Harris. *3264 and All That: A Second Course in Algebraic Geometry*. Cambridge University Press, 2016.

[FL94] William Fulton and Alain Lascoux. "A Pieri formula in the Grothendieck ring of a flag bundle". In: *Duke Mathematical Journal* 76.3 (1994), pp. 711–729.

[Ful91] William Fulton. "Flags, Schubert Polynomials, Degeneracy Loci, and Determinantal Formulas". In: *Duke Mathematical Journal* 65.3 (Mar. 1991), pp. 381–420.