Chow Groups and Classifying Spaces

Selected exercises from pcmi

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1 Exercise I: Chow Groups of the Circle

1.1 *Why Chow groups?*

Throughout this exposition, we fix a ground field $\boldsymbol{k}$, and by a scheme we mean a separated scheme of finite type over $\boldsymbol{k}$. A variety means an integral separated scheme of finite type over $\boldsymbol{k}$ (in particular, irreducible).

Homology and Cohomology have proven to be extremely useful as invariants of topological spaces but they require *extra structure* like chain complexes to work. The problem in AG is that our topologies are so different! It is for example impossible to run the CW machinery on varieties. And we cannot always pretend that we are over $\mathbb{C}$ to get the complex manifold structure, because a large part of AG is transferring our geometric intuition from $\mathbb{C}$ to other fields and characteristics and be able to go between them.

So we need to do (co)homology purely within the algebraic data. We want:

$$\begin{aligned} \text{topological cycles} &\rightsquigarrow \text{algebraic cycles} \\ \text{homologous cycles} &\rightsquigarrow \text{algebraic notion of equivalence} \end{aligned}$$

It turns out we have natural candidates for both. The first part is almost tautological:

> **Definition 1.1.** Let X be a scheme. The **group of algebraic cycles** $Z_\bullet(X)$ is the free abelian group on the set of closed *subvarieties* of X. $Z_\bullet(X)$ is graded by dimension:
> $$Z_\bullet(X) = \bigoplus_i Z_k(X),$$
> where $Z_k(X)$ is the subgroup of *dimension* k subvarieties.

Following [Ful84], we use square brackets to denote algebraic cycles coming from subvarieties, and greek letters for abstract cycles. In particular, if $\bullet = \dim X - 1$, then we recover the group of Weil divisors on X. Recall that for every non-zero rational function $f \in \boldsymbol{k}(X)^\times$, we may associate to it a Weil divisor by collecting its zeros and poles:

$$\operatorname{div} f := \sum_{\operatorname{codim} D=1} \operatorname{ord}_D f \cdot [D].$$

For more on order of vanishing, cf. [Ful84].

The second part is also straightforward, because we already have the notion of linear equivalence of divisors. Recall that two divisors C and D are linearly equivalent (we write $C \sim D$) if $[C] = \operatorname{div} f + [D]$ for some rational function f.

Definition 1.2. Two k-cycles α and β are **rationally equivalent**, written $\alpha \sim \beta$, if there are a finite number of $(k+1)$-dimensional subvarieties W_i of X, and rational functions $f_i \in \boldsymbol{k}(W_i)^\times$, such that

$$\alpha - \beta = \sum \operatorname{div}(f_i).$$

Algebraic k-cycles rationally equivalent to 0 form a subgroup of $Z_k(X)$, denoted $\operatorname{Rat}_k(X)$.

Definition 1.3. The k^{th} **Chow group** is defined as the group of k-cycles modulo rational equivalence.

$$CH_k(X) := Z_k(X)/\operatorname{Rat}_k(X) = Z_k(X)/\sim.$$

We can make the Chow groups into a direct sum graded by *dimension*:

$$CH_\bullet(X) = \bigoplus_k CH_k(X).$$

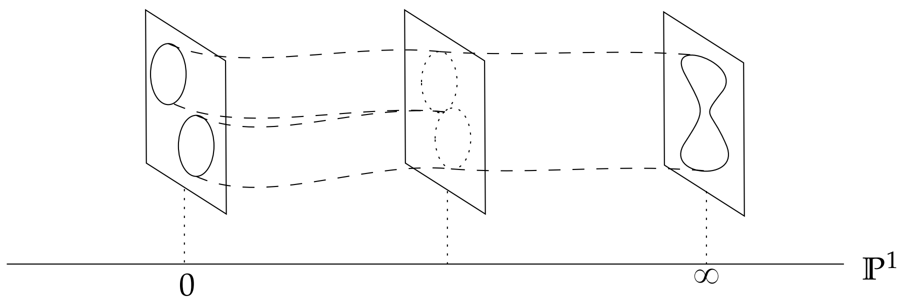
Okay, so rational equivalence looks like some beefed-up version of linear equivalence of divisors, which works well in proofs but does not provide much intuition. Thankfully, there is a more geometric way to think about rational equivalence.

Proposition 1.4 ([EH16] 1.2)**.** $\mathrm{Rat}(X) \subset Z(X)$ *is generated by differences of the form*

$$\langle \Phi \cap (\{t_0\} \times X) \rangle - \langle \Phi \cap (\{t_1\} \times X) \rangle,$$

where $t_0, t_1 \in \mathbb{P}^1$ *and* Φ *is a subvariety of* $\mathbb{P}^1 \times X$ *not contained in any vertical fiber* $\{t\} \times X$*.*

So rational equivalence is an algebraic version of homotopy/cobordism.



Just like cohomology, there is a natural product structure on Chow groups corresponding to intersection of cycles:

Theorem 1.5. *If X is a smooth quasi-projective variety, then there is a unique product structure on* $CH(X)$ *satisfying the condition:*

If two subvarieties A, B *of* X *are generically transverse, then*

$$[A][B] = [A \cap B].$$

This structure makes

$$CH^\bullet(X) = \bigoplus_{c=0}^{\dim X} CH^c(X)$$

into an associative, commutative ring, graded by codimension, called the ***Chow ring*** *of* X*.*

We use $CH_\bullet$ to denote Chow groups, and when X is smooth, we use $CH^\bullet$ to denote the Chow ring.

Remark 1.6. Chow groups are closely connected with the *Borel-Moore* homology groups, which is roughly the relative homology of the one-point compactification relative to infinity.

1.2 Useful properties of Chow groups

Every scheme X has a fundamental class $[X]$ which generates the top-dimension Chow group.

We can:

- push forward algebraic algebraic cycles under proper morphisms. (**proper pushforward**–"wrong way map," preserves dimension.)
- pull back divisor classes/line bundles always.
- pull back algebraic cycles under flat morphisms (**flat pullback**, preserves codimension).
- pull back algebraic cycles along *any* morphism between *smooth* schemes (**smooth pullback**, preserves codimension).

Theorem 1.7 (Excision). *If $Z \subseteq X$ is a closed subscheme, then the following right exact sequence, termed **excision/localization sequence**, holds for all degrees:*

$$CH_\bullet(Z) \to CH_\bullet(X) \to CH_\bullet(X \setminus Z) \to 0.$$

If X is smooth, then $CH^\bullet(X) \to CH^\bullet(X \setminus Z)$ is a ring homomorphism.

Proof

$Z \to X$ is proper, and $X \setminus Z \to X$ is flat. The above sequence follows from proper pushforward and flat pullback. ∎

Theorem 1.8 (Mayer-Vietoris). *If X_1, X_2 are closed subschemes of X, then there is a right exact sequence for all degrees:*

$$CH_\bullet(X_1 \cap X_2) \to CH_\bullet(X_1) \oplus CH_\bullet(X_2) \to CH_\bullet(X_1 \cup X_2) \to 0.$$

Example 1.9.

$CH^\bullet(\mathbb{A}^n) \cong \mathbb{Z}$. $CH^\bullet(\mathbb{P}^n) \cong \mathbb{Z}[\eta]/(\eta^{n+1})$, where η is the hyperplane class.

1.3 Chow groups of the circle

Now, let's compute the Chow groups of the circle! By the circle we mean the affine plane curve $S^1 \subseteq \mathbb{A}^2$ defined by $x^2 + y^2 = 1$, say over $\mathbb{R}$ and $\mathbb{C}$. Since S^1 has dimension 1, we only need to compute the Chow groups CH_0 and CH_1.

First, the top dimension CH_1 is just $\mathbb{Z} \cdot [S^1]$, so the interesting calculation is just the Chow group of points.

For CH_0, we remember that $S^1 \subseteq \mathbb{A}^2$ comes from the projective curves $C = Z[x^2 + y^2 + z^2] \subseteq \mathbb{P}^2$ with the point at infinity $p_\infty = Z[x^2 + y^2 + z^2, z] = Z[x^2 + y^2, z]$ removed.

Notice that C is a plane conic curve with a rational point over $\mathbb{R}$ or trivially $\mathbb{C}$, so $C \cong \mathbb{P}^1$ (the proof is simply projecting from the rational point and extending to an honest isomorphism).

Also for $p_\infty = Z[x^2 + y^2, z]$, since x, y, z cannot be all zero, and z is already zero at infinity, we must have $x, y \neq 0$. Therefore, we may elect to dehomogenize and set $y = 1$, so

$$p_\infty = \operatorname{Spec} k[x]/(x^2 + 1).$$

Now, let's use the localization sequence:

$$CH_0(p_\infty) \xrightarrow{\varphi} \underset{\mathbb{Z}}{CH_0(C)} \to CH_0(S^1) \to 0$$

$\underline{\boldsymbol{k} = \mathbb{R}}$

Over the reals, $p_\infty = \operatorname{Spec} \mathbb{R}[x]/(x^2+1) \cong \operatorname{Spec} \mathbb{C}$, and the RES becomes

$$\mathbb{Z} \xrightarrow{\varphi} \mathbb{Z} \to CH_0(S^1_{\mathbb{R}}) \to 0,$$

where the homomorphism φ has the form $(\cdot d)$. We claim that $d = 2$. This can be seen from either of the following:

- The degree map $\deg : CH_0(p_\infty) \cong \underset{\mathbb{Z}}{CH_0(\operatorname{Spec} \mathbb{C})} \to \underset{\mathbb{Z}}{CH_0(\operatorname{Spec} \mathbb{R})}$ is given by the degree of the field extension $[\mathbb{C} : \mathbb{R}] = 2$, and it factors as

$$\begin{array}{ccc} CH_0(p_\infty) & \xrightarrow{\varphi} & CH_0(C_{\mathbb{R}}) \\ & {\scriptstyle \deg = (\cdot 2)} \searrow & \downarrow \cong \\ & & CH_0(\operatorname{Spec} \mathbb{R}) \end{array}$$

  Therefore, φ is multiplication by 2.

- More concretely, the rational function $f(x) = x^2 + 1$ on $\mathbb{P}^1_{\mathbb{R}}$ has one zero at $(x^2+1) = p_\infty$ and two poles at $\infty \cong \operatorname{Spec} \mathbb{R}$, so

$$0 \sim \operatorname{div}(f) = p_\infty - 2(\operatorname{Spec} \mathbb{R}).$$

$CH_0(S^1_{\mathbb{R}})$ is the cokernel of φ, hence isomorphic to $\mathbb{Z}/2$.

$$CH_\bullet(S^1_{\mathbb{R}}) \cong \begin{cases} \mathbb{Z}/2 & \bullet = 0 \\ \mathbb{Z} & \bullet = 1. \end{cases}$$

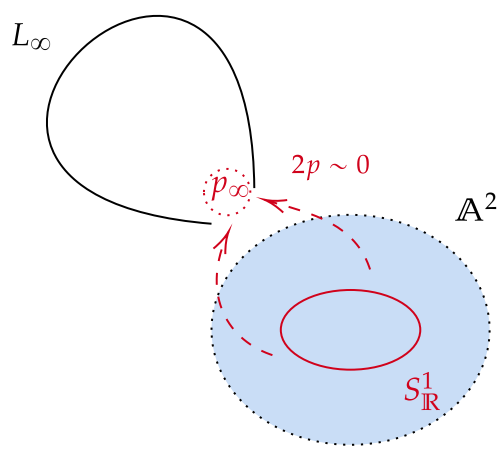
$\underline{\boldsymbol{k} = \mathbb{C}}$

Over the complex numbers, $p_\infty = \operatorname{Spec} \mathbb{C}[x]/(x^2+1) \cong \operatorname{Spec} \mathbb{C} \sqcup \operatorname{Spec} \mathbb{C}$, so the RES becomes

$$CH_0(\operatorname{Spec} \mathbb{C} \sqcup \operatorname{Spec} \mathbb{C}) \cong \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\varphi} \mathbb{Z} \to CH_0(S^1_{\mathbb{C}}) \to 0$$

where φ is the map $(x, y) \mapsto x + y$, a surjection. Hence, $CH_0(S^1_{\mathbb{C}}) = \operatorname{coker} \varphi \cong 0$.

$$CH_\bullet(S^1_{\mathbb{C}}) \cong \begin{cases} 0 & \bullet = 0 \\ \mathbb{Z} & \bullet = 1. \end{cases}$$



1.4 Chow groups of affine plane minus a conic

Now, let's try to compute the Chow groups of the affine plane $\mathbb{A}^2$ minus a plane conic C.

Let $U = \mathbb{P}^2 \backslash C$. First, since U is irreducible,

$$CH_2(U) \cong \mathbb{Z} \cdot [U].$$

Also, we know that $C \cong \mathbb{P}^1_{\mathbb{C}}$, so we know that $CH_0(C), CH_1(C) \cong \mathbb{Z}$, while the rest are all 0. We have the localization sequence

$$\underset{\mathbb{Z}}{CH_1(C)} \xrightarrow{\iota_*} \underset{\mathbb{Z}}{CH_1(\mathbb{P}^2)} \to CH_1(U) \to 0.$$

The closed immersion $\iota : C \to \mathbb{P}^2$ is of degree 2, so the proper pushforward ι_* just sends $[C]$ to twice the hyperplane $2[H_{\mathbb{P}^2}]$, which is multiplication by 2. Hence the sequence reads

$$\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to CH^1(U) \to 0$$

which implies that

$$CH^1(U) \cong \mathbb{Z}/2.$$

Lastly,

$$\underset{\mathbb{Z}}{CH_0(C)} \xrightarrow{\cdot 1} \underset{\mathbb{Z}}{CH_0(\mathbb{P}^2)} \to CH_0(U) \to 0$$

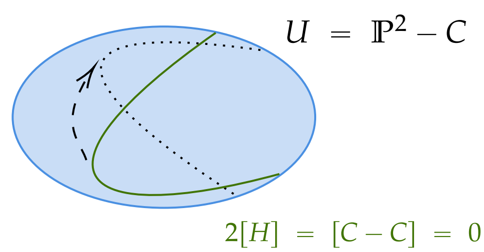
implies

$$CH_0(U) = 0.$$

Therefore,

$$CH_\bullet(U) \cong \begin{cases} 0 & \bullet = 0 \\ \mathbb{Z}/2 & \bullet = 1 \\ \mathbb{Z} & \bullet = 2. \end{cases}$$

The fact that $CH_1(U)$ is torsion can be seen geometrically. $CH_1(U)$ is generated by the restriction of the hyperplane class $[H]$ from $\mathbb{P}^2$. In $\mathbb{P}^2$, $2[H]$ is the class of any conic, which is rationally equivalent to given conic C that we removed from U. Therefore, $2[H] \sim [C \setminus C] = 0$ on U.



2 Exercise II: Chow Groups of A Quotient Singularity

2.1 More properties of Chow groups

$f : X \to Y$ is a locally-trivial $\mathbb{A}^r$-bundle if Y is covered by open subsets U such that $f^{-1}(U)$ is isomorphic to $U \times \mathbb{A}^r$ over U.

Theorem 2.1 (Homotopy invariance of Chow groups). *The pullback along a locally trivial $\mathbb{A}^r$-bundle f induces an isomorphism of Chow groups*

$$f^* : CH_i(Y) \xrightarrow{\cong} CH_{i+r}(X).$$

If Y is furthermore smooth ($\implies$ X is also smooth), then we can use upper-indexing to make it easier to remember:

$$f^* : CH^i(Y) \xrightarrow{\cong} CH^i(X).$$

Next, more results about Chow rings of affine and projective bundles.

Theorem 2.2 ([EH16] Theorem 9.6)**.** ***Projective bundle formula.*** *Let $\mathcal{E}$ be a vector bundle of rank $r+1$ on a smooth projective scheme X, and let $\zeta = c_1\left(\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)\right) \in CH^1(\mathbb{P}\mathcal{E})$. Let $\pi : \mathbb{P}\mathcal{E} \to X$ be the projection. The map $\pi^* : CH(X) \to CH(\mathbb{P}\mathcal{E})$ is an injection of rings, and via this map*

$$CH(\mathbb{P}\mathcal{E}) \cong CH(X)[\zeta]/\left(\zeta^{r+1} + c_1(\mathcal{E})\zeta^r + \cdots + c_{r+1}(\mathcal{E})\right)$$

In particular, the group homomorphism $CH(X)^{\oplus r+1} \to CH(\mathbb{P}\mathcal{E})$ given by $(\alpha_0, \ldots, \alpha_r) \mapsto \sum \zeta^i \pi^*\left(\alpha_i\right)$ is an isomorphism, so that

$$CH(\mathbb{P}\mathcal{E}) \cong \bigoplus_{i=0}^{r} \zeta^i CH(X)$$

as groups.

Let L be a line bundle over a smooth scheme X, viewed as the total space of an invertible sheaf $\mathcal{L}$ with a projection $L \to X$ with fibers $\mathbb{A}^1$. Let $\sigma : X \to L$ be the zero section of L. We can describe the Chow ring of $L -$ zero section with the first Chern class of L:

Proposition 2.3. *Let L be a line bundle over a smooth scheme X. Then,*

$$CH^\bullet(L - \sigma(X)) \cong CH^\bullet(X)/\left(c_1(L)\right).$$

Proof

Use excision. The inclusion of the zero section into L induces a proper pushforward on Chow groups

$$CH^{\bullet-1}(L) \to CH^\bullet(L)$$

given multiplication by the first Chern class $c_1(L)$. The localization sequence

$$CH^{\bullet-1}(L) \to CH^\bullet(L) \to CH^\bullet(L - \sigma(X)) \to 0$$

then gives the isomorphism

$$CH^\bullet(L - \sigma(X)) \cong CH^\bullet(X)/\left(c_1(L)\right).$$

■

More generally, let E be a vector bundle of rank r on a smooth scheme X over a field k. We can compute the Chow ring of the total space of E minus the zero-section (isomorphic to X):

Proposition 2.4 (Chow ring of an $\mathbb{A}^r - O$-bundle).

$$CH^\bullet(E - X) \cong CH^\bullet(X)/\left(c_r(E)\right).$$

Proof

Use the self-intersection formula together with the localization sequence. ■

2.2 Rational surface singularity of type A_2

The rational surface singularity of type A_2 locally looks like the quotient of the affine plane $\mathbb{A}^2$ by the cyclic group $C_3 = \{1, w, w^2\}$, with the generator w acting as

$$w \cdot (x, y) = (\zeta_3 x, \zeta_3^{-1} y),$$

where ζ_3 is the third root of unity.

Let $X = \mathbb{A}^2/C_3$ denote this geometric quotient. Since X is the quotient of an affine scheme by an affine algebraic group, the following correspondence applies:

G-quotient of an affine scheme $\stackrel{\sim}{\longleftrightarrow}$ subring of G-invariants in the coordinate algebra

Let's use this to compute the variety structure of X. We know that $\mathcal{O}_X(X)$ is the subring in $\boldsymbol{k}[x, y]$ of C_3-invariants. It suffices to find all monomials $x^a y^b$ such that

$$(\zeta_3 x)^a (\zeta_3^{-1} y)^b = x^a y^b.$$

This implies that

$$\zeta_3^{a-b} = 1,$$

which means

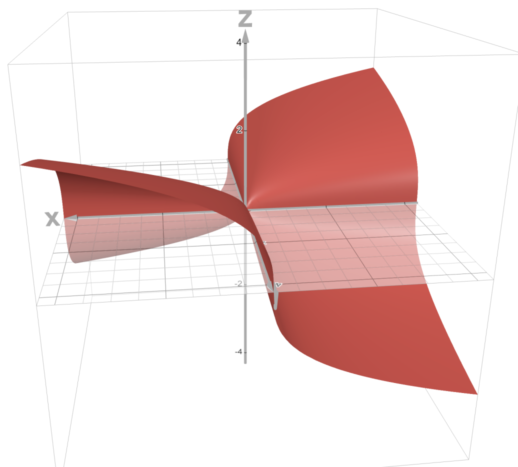
$$a \equiv b \mod 3.$$

Plotting this in the lattice $\mathbb{Z} \times \mathbb{Z}$, we find that these are generated by the monomials

x^3, xy, y^3. Hence,

$$\boldsymbol{k}[x,y]^{C_3} \cong \boldsymbol{k}[x^3, xy, y^3] \cong \boldsymbol{k}[\alpha, \beta, \gamma]/(\beta^3 - \alpha\gamma).$$

Hence, X is a surface in $\mathbb{A}^3$ which is singular at the origin.



2.3 Chow groups of A_2-surface singularity

There are people who want to do intersection theory over such singular spaces, and they ask us to compute its Chow groups. Namely, we are given the task to compute

$$CH_\bullet(X), \quad \bullet = 0, 1, 2.$$

<u>CH_2</u>

The easiest one. Since X is irreducible, the top Chow group is isomorphic to $\mathbb{Z}$ generated by the fundamental class.

$\underline{CH_0}$

The next easiest. Our strategy is to use localization to single out the singular point $O \in X$. The localization sequence for O reads

$$CH_0(O) \to CH_0(X) \to \underset{\cong 0}{CH_0((\mathbb{A}^2 - 0)/C_3)} \to 0$$

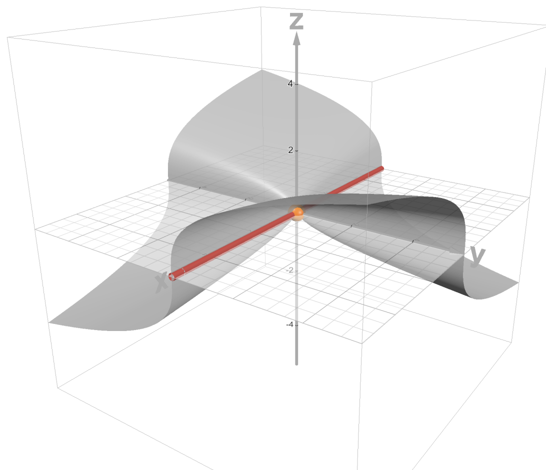
which implies that $CH_0(X)$ is generated by the class $[O]$. Now, it is easier if we embed X as the singular surface $\operatorname{Spec} \boldsymbol{k}[\alpha, \beta, \gamma]/(\beta^3 - \alpha\gamma) \subseteq \mathbb{A}^3$. There are closed embeddings

$$0 \hookrightarrow \mathbb{A}^1 \cong x\text{ - axis} \hookrightarrow X$$

and the proper pushforward of $CH_0(O)$ factors as

$$\begin{array}{ccc} CH_0(O) & \longrightarrow & CH_0(X) \\ & \searrow \qquad \nearrow & \\ & \underset{\cong 0}{CH_0(x\text{ - axis})} & \end{array}$$

yielding $CH_0(X) = 0$.



O lying on x-axis within X.

$\underline{CH_1}$

The tricky case. $CH_1(X)$ coincides with the Picard group of X, which can be very tricky to compute on singular varieties. We also understand that the origin of all our troubles is the fixed point O – i.e. X is singular because the C_3-action is not free!

Luckily, we have a secret weapon up our sleeves:

Lemma 2.5. *Let G be a group, E be a free G-set. Let X be any other G-set. Then, the product $E \times X$ with coordinate-wise action is a free G-set, even though the action on X might not be free.*

Definition 2.6. Let E be a free G-space and X be any G-space. The **associated fiber bundle** is defined as

$$E \times_G X := (E \times X)/G.$$

> It is a G-fiber bundle over the base $B = E/G$ with fibers isomorphic to X.

So, let's find some space with a free C_3-action and use it to give some extra room for C_3 to move around on $\mathbb{A}^2$. To be completely safe, let's choose $E = \mathbb{A}^N - O$ for some large N, and let $X' := E \times_{C_3} \mathbb{A}^2 = (E \times \mathbb{A}^2)/C_3$. X' is a $\mathbb{A}^2$-bundle over the base $B = (\mathbb{A}^N - O)/C_3$ with an C_3-action on the fibers. The zero section $\sigma : B \to X'$ sends a point $b \in B$ to the origin $\{b\} \times \{O\}$ in the fiber $\{b\} \times \mathbb{A}^2$.

Since all actions are free now, we expect every quotient in the above construction to be *smooth*. This justifies switching to a Chow-ring-style grading by *codimension*, which is what we will do from now on. The reason will become clear when we reveal the true nature of this construction.

We can form a localization sequence by removing the image of this zero section from X':

$$CH^1(B) \to CH^1(X') \to CH^1(X' \setminus \sigma(B)) \to 0.$$

To proceed, let's do a term-by-term calculation.

$\boxed{CH^\bullet(B)}$

Since the $\mathbb{G}_m$-action on $\mathbb{A}^N - O$ is free, and the C_3-action is a subrepresentation, we have the following "fiber sequence."

$$\mathbb{G}_m/C_3 \to (\mathbb{A}^N - O)/C_3 \to (\mathbb{A}^N - O)/\mathbb{G}_m.$$

The first and third terms of this sequence are familiar spaces: $\mathbb{A}^1 - 0 \cong \mathbb{G}_m/C_3$, and $(\mathbb{A}^N - O)/\mathbb{G}_m \cong \mathbb{P}^{N-1}$. In equivalent terms, we have a fiber sequence

$$\mathbb{A}^1 - 0 \to B \xrightarrow{\pi} \mathbb{P}^{N-1},$$

realizing B as a line bundle over $\mathbb{P}^{N-1}$ minus the zero section. *Which line bundle is it?*

Calculating $\mathcal{O}_B$ over open sets shows that $H^0(B, \mathcal{O}_B)$ consists of all cubic forms over $\mathbb{P}^{N-1}$, so we see that B is the total space of $\mathcal{O}(3)$ minus the zero section. *Is there a slicker argument?*

By Chow ring of line bundle minus the zero section (Proposition 2.3), the Chow ring of B is given by

$$CH^\bullet(B) \cong CH^\bullet(\mathbb{P}^{N-1})/c_1(\mathcal{O}(3)) \cong \mathbb{Z}[\eta]/(3\eta, \eta^N)$$

where η is the class of a hyperplane in $\mathbb{P}^{N-1}$.

$\boxed{CH^3(B) \xrightarrow{\varphi} CH^1(X')}$

Since X' is an $\mathbb{A}^2$-bundle over B with action of $\mathbb{G}_m$ of bidegree $(1,-1)$ on the fibers, the map φ is multiplication by the second Chern class $c_2(\mathcal{O}(1) \oplus \mathcal{O}(-1)) = \eta \cdot (-\eta) = -\eta^2$. This induces an isomorphism for the third term

$$CH^1(X' \smallsetminus \sigma(B)) \cong CH^3(B) \cong \mathbb{Z}/3.$$

But what is this third term?

$\boxed{CH^1(X' \smallsetminus \sigma(B))}$

$X' \smallsetminus \sigma(B) = \left((\mathbb{A}^N - O) \times (\mathbb{A}^2 - O)\right)/C_3$ – there are some complicated excisions. We need to use the following property of Chow groups:

Lemma 2.7 (Key lemma for well-definition of classifying spaces). *Let V_1, V_2 be two G-representations, with $S_1 \subseteq V_1, S_2 \subseteq V_2$ closed locus of codimension greater than i, such that the G-actions on $V_1 \smallsetminus S_2$ and $V_2 \smallsetminus S_2$ are free. Then,*

$$CH^i\left((V_1 \smallsetminus S_1)/G\right) \cong CH^i\left((V_2 \smallsetminus S_2)/G\right)$$

Proof

We have two bundles

$$V_2 \to \left((V_1 - S_1) \times V_2\right)/G \to (V_1 - S_1)/G$$

and

$$V_1 \to (V_1 \times (V_2 - S_2))/G \to (V_2 - S_2)/G.$$

Remark [Tot24] In fact, these are both vector bundles for the flat topology; but, by Hilbert's Theorem 90 or Grothendieck's theory of faithfully flat descent, vector bundles in the flat topology are the same as the usual notion of vector bundles in the Zariski topology.

By homotopy invariance of Chow groups (Theorem 2.1), it suffices to check that $(V_1 - S_1) \times V_2/G$ has the same Chow groups as $V_1 \times (V_2 - S_2)/G$. Since we assumed that $\operatorname{codim} S_1, \operatorname{codim} S_2 > i$, the localization sequences produce isomorphisms

$$\begin{aligned} CH^i\left(((V_1 - S_1) \times V_2)/G\right) &\cong CH^i\left(((V_1 - S_1) \times (V_2 - S_2))/G\right) \\ &\cong CH^i\left((V_1 \times (V_2 - S_2))/G\right) \end{aligned}$$

as desired.

■

By the above lemma, $CH^1(X' \setminus \sigma(B)) \cong CH^1((\mathbb{A}^2 - O)/C_3)$, so

$$CH^1((\mathbb{A}^2 - O)/C_3) \cong \mathbb{Z}/3.$$

One final localization sequence removing O from $X = \mathbb{A}^2/C_3$ gives us

$$\underbrace{CH^1(O)}_{=0} \to CH^1(X) \to CH^1\left((\mathbb{A}^2 - O)/C_3\right) \to 0$$

whence we may finally conclude that $CH^1(X) = \mathbb{Z}/3$.

$$CH_\bullet(X) = \begin{cases} 0 & \bullet = 0 \\ \mathbb{Z}/3 & \bullet = 1 \\ \mathbb{Z} & \bullet = 2. \end{cases}$$

3 Exercise III: Chow Groups of Classifying Spaces and Equivariant Intersection Theory

Summarizing what just happened: we took a G-representation V which became free after removing a closed set S of sufficiently high codimension. Then, we formed the associated bundle $(V - S) \times_G X$ whose Chow groups are governed by the base $B = (V - S)/G$. We can package this information into a more compact theory: the scheme B is what lies behind *classifying spaces*, and the Chow groups we computed are the essence of *equivariant Chow groups*. These computations and associated investigations are what's called *equivariant intersection theory*.

3.1 Classifying spaces for algebraic geometers

Classifying spaces in algebraic geometry and Chow rings thereof were first introduced by Burt Totaro in [Tot98].

> **Definition 3.1.** Let X be a scheme with an action by an affine group scheme G over a field $\boldsymbol{k}$. Let i be an integer. Let V_i be a representation of G such that G acts freely on a closed subset $S_i \subset V_i$ with $\operatorname{codim}(S_i \subset V_i) > \dim(X) - i$. Any such V_i is called an **approximation space** of degree i.

> **Definition 3.2.** Let $\{(S_i \subset V_i)\}$ be a collection of approximation spaces for the affine group scheme G, one in each degree i. A **universal bundle** EG is the collection of schemes
>
> $$EG = \{V_0 - S_0, V_1 - S_1, V_2 - S_2, \dots\}.$$
>
> Let $B_i = \frac{V_i - S_i}{G}$. A **classifying space** for G, denoted BG, is the collection of schemes
>
> $$BG = \{B_0, B_1, B_2, \dots\}.$$

The **Chow groups of a classifying space** BG is defined as

$$CH^i(BG) := CH^i(B_i) = CH^i((V_i - S_i)/G).$$

By the key lemma 2.7, the Chow groups $CH^\bullet(BG)$ are well-defined regardless of the choice of approximation spaces V_i. We say that EG and BG are *well-defined up to homotopy.*

Example 3.3. ([Tot24] 2.2.2) Consider the multiplicative group $\mathbb{G}_m$ over a field k. $\mathbb{G}_m$ has an obvious faithful representation W of dimension 1, namely $G_m \xrightarrow{\cong} GL(1)$. Taking the direct sum of $N+1$ copies of W gives a representation on which G_m acts freely outside the origin by

$$t \cdot (x_0, \dots, x_n) = (tx_0, \dots, tx_n).$$

Therefore, for $0 \leqslant i \leqslant N$, we may take $W^{\oplus(N+1)}$ to be an approximation space for $B\mathbb{G}_m$ and compute:

$$\begin{aligned} CH^i(B\mathbb{G}_m) &\cong CH^i\left(\left(\mathbb{A}^{N+1} - 0\right)/\mathbb{G}_m\right) \\ &\cong CH^i(\mathbb{P}^N) \\ &\cong \mathbb{Z}. \end{aligned}$$

Moreover, the Chow ring of $B\mathbb{G}_m$ is defined to agree with the Chow ring of $\mathbb{P}^N$ in degrees at most N, and so we have

$$CH^\bullet B\mathbb{G}_m \cong \mathbb{Z}[\eta]$$

where the "hyperplane class" $|\eta| = 1$ (meaning that $\eta \in CH^1(B\mathbb{G}_m)$).

Example 3.4. From the previous section, we have seen that over $\boldsymbol{k} = \mathbb{C}$,

$$CH^\bullet(BC_3) \cong \mathbb{Z}[\eta]/(3\eta), \quad |\eta| = 1.$$

Similarly, for p prime, we have

$$CH^{\bullet}(BC_p) \cong \mathbb{Z}[\eta]/(p\eta), \quad |\eta| = 1.$$

Example 3.5. ([Tot24] 2.2.4) The Chow ring of BGL_n over any field k is the polynomial ring

$$CH^*(BGL_n) = \mathbb{Z}[c_1, \ldots, c_n]$$

with $|c_i| = i$. These generators are called *universal Chern classes*.

3.2 Equivariant Chow groups

Definition 3.6. Let X be a scheme with an action by an affine group scheme G over a field $\boldsymbol{k}$. Let i be an integer. Let $(S_i \subset V_i)$ be approximation spaces for G. The **Borel construction/homotopy quotient**, denoted $EG \times_G X$ or resp. $X//G$, is the collection of schemes

$$((V_0 - S_0) \times X)/G, ((V_1 - S_1) \times X)/G, ((V_2 - S_2) \times X)/G, \ldots$$

Definition 3.7. Let X, G, V_i as above. We define the i th **equivariant Chow group** by

$$CH_i^G(X) = CH_{i+\dim(V_i)-\dim(G)}(((V_i - S_i) \times X)/G)$$

By the key lemma 2.7, these groups are independent of the choice $S \subset V$. Since $CH_G^{\bullet}$ is defined as the Chow ring of certain smooth schemes, usual properties like proper pushforward, smooth pullback, excision, etc. hold for $CH_G^{\bullet}$ too.

Definition 3.8. If in addition X is smooth, then we use the codimension grading and define $CH_G^i(X) = CH^G_{\dim(X)-i}(X)$, and $CH_G^\bullet(X)$ is a commutative graded ring. The above definition becomes easier to remember if we use the grading by codimension:

$$CH_G^i(X) = CH^i(((V_i - S_i) \times X)/G).$$

Example 3.9. Equivariant Chow groups of a point. We have

$$CH_G^\bullet(\operatorname{Spec} \boldsymbol{k}) = CH^\bullet(BG).$$

If X is a smooth $\boldsymbol{k}$-scheme with a G-action, then the smooth pullback along the structural map $X \to \operatorname{Spec} \boldsymbol{k}$ induces a ring homomorphism

$$CH_G^\bullet(\operatorname{Spec} \boldsymbol{k}) \to CH_G^\bullet(X)$$

making $CH_G^\bullet(X)$ a $CH^\bullet(BG)$-module.

Example 3.10. If X is smooth and G acts freely on X, then

$$CH_G^\bullet(X) \cong CH^\bullet(X/G).$$

To proof this, use homotopy invariance of Chow groups.

3.3 Chow groups of the affine line with two origins

Let $\mathbb{G}_m$ act on the affine plane $\mathbb{A}^2$ over $\boldsymbol{k}$ by

$$t \cdot (x, y) = \left(tx, t^{-1}y\right).$$

This action is free outside of the origin O, so we can use equivariant Chow groups to calculate the Chow ring of the quotient $(\mathbb{A}^2 - O)/\mathbb{G}_m$. To do this, we use the localization sequence:

$$CH^\bullet_{\mathbb{G}_m}(pt) \to CH^\bullet_{\mathbb{G}_m}(\mathbb{A}^2) \to CH^\bullet_{\mathbb{G}_m}(\mathbb{A}^2 - O) \to 0.$$

We have seen that

$$CH^\bullet_{\mathbb{G}_m}(pt) = CH^\bullet(B\mathbb{G}_m) \cong \mathbb{Z}[u],$$

where $|u| = 1$ is the hyperplane class of $\mathbb{P}^N$ for N sufficiently large, i.e $u = c_1\mathcal{O}(1)$. Also, notice that

$$CH^\bullet_{\mathbb{G}_m}(\mathbb{A}^2) := CH^\bullet(\mathbb{A}^2 \times_{\mathbb{G}_m} E\mathbb{G}_m)$$

is the associated fiber bundle to the universal bundle $E\mathbb{G}_m \to B\mathbb{G}_m$ with fiber $\mathbb{A}^2$, so it is a rank 2 vector bundle over the base $B\mathbb{G}_m$. By *homotopy invariance*, we have

$$CH^\bullet(\mathbb{A}^2 \times_{\mathbb{G}_m} E\mathbb{G}_m) \cong CH^{\bullet+2}(B\mathbb{G}_m).$$

Then, the third term $CH^\bullet_{\mathbb{G}_m}(\mathbb{A}^2 - 0)$ can be viewed as the Chow group of the bundle $\mathbb{A}^2 \times_{\mathbb{G}_m} E\mathbb{G}_m$ minus the zero section. The map $CH^\bullet(B\mathbb{G}_m) \to CH^\bullet(\mathbb{A}^2 \times_{\mathbb{G}_m} E\mathbb{G}_m)$ is multiplication by the second Chern class, and one can prove using the localization sequence and the self-intersection formula that

$$CH^\bullet(\mathbb{A}^2 \times_{\mathbb{G}_m} E\mathbb{G}_m - \text{ zero section }) \cong CH^\bullet(B\mathbb{G}_m)/c_2(\mathbb{A}^2 \times_{\mathbb{G}_m} E\mathbb{G}_m).$$

To compute c_2, we notice that the action $t \cdot (x, y) = (t^1 x, t^{-1} y)$ corresponds to a direct sum of 1-dimensional faithful representations $\mathcal{O}(1) \oplus \mathcal{O}(-1)$, so it has total Chern class

$$c(\mathcal{O}(1) \oplus \mathcal{O}(-1)) = (1 + u)(1 - u) = 1 - u^2,$$

from which we read off

$$c_2 = -u^2.$$

Hence,

$$CH^\bullet(\frac{\mathbb{A}^2 - O}{\mathbb{G}_m}) \cong \mathbb{Z}[u]/(-u^2) \cong \begin{cases} \mathbb{Z} & \bullet = 0 \\ \mathbb{Z} & \bullet = 1 \\ 0 & \bullet = 2. \end{cases}$$

Okay, but what is this space really? To compute the geometric quotient $(\mathbb{A}^2 - O)/\mathbb{G}_m$, we cover $\mathbb{A}^2 - O$ with open sets $U_x := D(x) = \operatorname{Spec} k[x^{\pm 1}, y]$ and $U_y := D(y) = \operatorname{Spec} k[x, y^{\pm 1}]$. On U_x, the $\mathbb{G}_m$-invariants $k[x^{\pm 1}, y]^{\mathbb{G}_m}$ are rational functions in the form $\frac{f(x,y)}{y^r}$ wuch that

$$\frac{f(tx, t^{-1}y)}{t^{-r}y^r} = \frac{f(x,y)}{y^r}.$$

This means that each monomial $x^a y^b$ in f must have its exponents satisfying $b - a = r$, which means that $f(x,y)/y^r$ is really a polynomial in x/y! Thus, we have

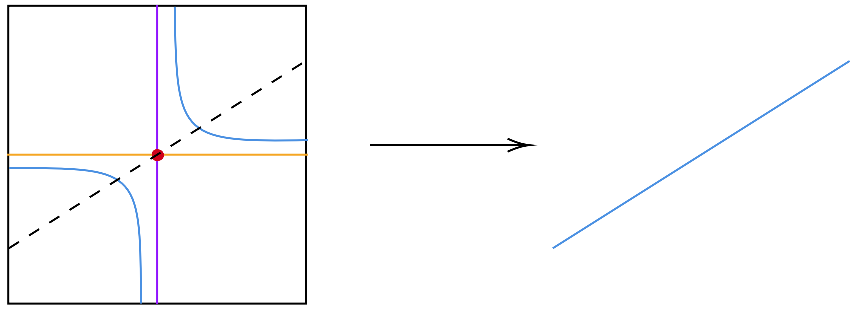
$$U_x/\mathbb{G}_m = \operatorname{Spec} k[x/y] \cong \mathbb{A}^1.$$

Similarly, one checks that

$$U_y/\mathbb{G}_m = \operatorname{Spec} k[x/y] \cong \mathbb{A}^1,$$

with transition function given by the identity $x/y \mapsto x/y$. Hence, $(\mathbb{A}^2 - O)/\mathbb{G}_m$ is just the affine line with two origins.

Geometrically, there are 4 kinds of orbits: non-degenerate hyperbola, the x, y-axes, and the origin. So fix any slanted line through the origin. In the quotient, exactly one hyperbola lies over each point away from the origin, but the two axes both lie over the origin, so we get the double point there.



Oops, we have just calculated the Chow ring of non-separated scheme. Also, it appears to have the same Chow ring as the other way of gluing two $\mathbb{A}^1$'s, namely $\mathbb{P}^1 \cong (\mathbb{A}^2 - O)/\mathbb{G}_m$ where the action is the standard one with bidegree $(1,1)$. Let that sink in for a minute.

$$\begin{array}{ccc} \mathbb{A}^1 - O & \longrightarrow & \mathbb{A}^1 \\ \downarrow & & \downarrow \\ \mathbb{A}^1 & \longrightarrow & \mathbb{A}^1 \text{ with two origins} \end{array} \qquad \begin{array}{ccc} \mathbb{A}^1 - O & \longrightarrow & \mathbb{A}^1 \\ \downarrow & & \downarrow \\ \mathbb{A}^1 & \longrightarrow & \mathbb{P}^1 \end{array}$$

Remark 3.11. [Tot24] Edidin and Graham show that the usual definition of Chow groups actually works, with the usual properties, for all algebraic spaces of finite type over a field ([EG97] §6.1). This includes all schemes of finite type over a field, separated or not.

3.4 Classifying spaces in motivic homotopy theory

Our current definitions of EG and BG feels clumsy, because these infinite collections aren't really *spaces*. In homotopy theory (i.e. over $\mathbb{R}$ or $\mathbb{C}$), BG is defined to be an infinite colimit of CW-complexes which is well-defined up to homotopy. Under the algebraic setting, even if we often get nested spaces

$$BG = \{B_0 \subseteq B_1 \subseteq B_2 \subseteq \cdots\}$$

from the approximation spaces, the problem is that there isn't a single infinite-dimensional scheme that is the colimit of this chain of inclusions – the category of smooth $\boldsymbol{k}$-schemes Sm_k *doesn't have enough colimits*. This is the same reason why we cannot take arbitrary quotients of schemes.

One solution to this problem is to bump up Sm_k to a category of presheaves over it. *We can't always glue schemes together, but we can always glue presheaves*. For homotopical reasons, we consider presheaves valued in simplicial sets. For geometric reasons, we ask our presheaves to satisfy *Nisnevich descent*. We use $\mathrm{PSh}(\mathrm{Sm}_k)$ to denote the category of simplicial presheaves over Sm_k and use $\mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_k)$ to denote the full subcategory of Nisnevich sheaves. By localization in category theory, this is equivalent to the data $L_{\mathrm{Nis}}\,\mathrm{PSh}(\mathrm{Sm}_k)$ where L_{Nis} is the left adjoint to the inclusion $\mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_k) \subset \mathrm{PSh}(\mathrm{Sm}_k)$ called **Nisnevich localization**.

Also, we observe that Chow groups, like many other algebraic invariants, are homotopy invariant (invariant w.r.t. $\mathbb{A}^1$-bundles). So we would like to take the full subcategory $\mathrm{PSh}_{\mathbb{A}^1}(\mathrm{Sm}_k) \subset \mathrm{PSh}(\mathrm{Sm}_k)$ of $\mathbb{A}^1$-invariant simplicial presheaves. This is again equivalent to the left adjoint functor $L_{\mathbb{A}^1}$ called $\mathbb{A}^1$**-localization**.

Combining these together, we construct the following category where it makes sense to talk about homotopy theory of schemes:

Definition 3.12. The category of **motivic spaces**, denoted Spc_k, is the full subcategory in $\mathrm{PSh}(\mathrm{Sm}_k)$ of $\mathbb{A}^1$-invariant Nisnevich sheaves

$$\mathrm{Spc}_k := L_{\mathbb{A}^1} L_{\mathrm{Nis}}\,\mathrm{PSh}(\mathrm{Sm}_k)$$

Now, the classifying spaces EG, BG naturally live inside of Spc_k as motivic spaces over $\boldsymbol{k}$. Furthermore, we can make sense of the fact that the Chow ring of the affine line with two origins (let's denote it by $\div$) agrees with the Chow ring of $\mathbb{P}^1$:

Recall that we have two cartesian diagrams

$$\begin{array}{ccc} \mathbb{A}^1 - O & \longrightarrow & \mathbb{A}^1 \\ \downarrow & & \downarrow \\ \mathbb{A}^1 & \longrightarrow & \div \end{array} \qquad\qquad \begin{array}{ccc} \mathbb{A}^1 - O & \longrightarrow & \mathbb{A}^1 \\ \downarrow & & \downarrow \\ \mathbb{A}^1 & \longrightarrow & \mathbb{P}^1 \end{array}$$

in the category of k-schemes. These schemes naturally sit inside Spc_k via the Yoneda embedding, but we have inverted the $\mathbb{A}^1$'s in Spc_k, so they become *contractible* (i.e. homotopy equivalent to a point!) So we see that these two diagrams are equivalent to

$$\begin{array}{ccc} \mathbb{G}_m & \longrightarrow & * \\ \downarrow & \ulcorner & \downarrow \\ * & \longrightarrow & \div \end{array} \qquad\qquad \begin{array}{ccc} \mathbb{G}_m & \longrightarrow & * \\ \downarrow & \ulcorner & \downarrow \\ * & \longrightarrow & \mathbb{P}^1 \end{array}$$

which are both by definition *the suspension of* $\mathbb{G}_m$, denoted $\Sigma\mathbb{G}_m$. Thus, homotopy equivalent motivic spaces have isomorphic Chow groups.

For more on motivic homotopy theory, have a look at this very nice set of notes by Thomas Brazelton [Bra24] and also the original paper by Fabien Morel and Vladimir Voevodsky [MV99].

This setup allows us to talk about classifying spaces for groups over arbitrary fields, and we can try relating these groups to cohomology. Recall that there is a *cycle class map* from ordinary Chow groups to cohomology, that is in general neither injective nor surjective. A similar story happens for Chow groups of BG.

Example 3.13. For $G = (\mathbb{Z}/p)^n$ an elementary abelian group, the integral Chow ring is

$$CH^\bullet B(\mathbb{Z}/p)^n = \mathbb{Z}\left[y_1, \ldots, y_n\right] / \left(py_1, \ldots, py_n\right).$$

The mod-p Chow ring is then

$$CH^\bullet\left(B(\mathbb{Z}/p)^n\right)/p = \mathbb{F}_p\left[y_1, \ldots, y_n\right], \quad |y_i| = 1,$$

where as the group cohomology with $\mathbb{F}_p$ coefficients is

$$H^\bullet(B(\mathbb{Z}/p)^n; \mathbb{F}_p) = \mathbb{F}_p\langle x_1, \ldots, x_n, y_1, \ldots, y_n\rangle, \quad |x_i| = 1, |y_i| = 2.$$

Heuristically, the cycle class map only hits the algebraic part generated by the y_i (notice also the doubling in degrees.)

There are even examples where $CH^\bullet BG$ is not finitely generated (cf. [Tot24] §4 for many interesting calculations; cf. also the original paper [Tot98] and the fabulour book [Tot14]).

Luckily, we still have some control over the degree of the generators:

Proposition 3.14 ([Tot24] Theorem 4.1.1)**.** *For an affine group scheme G over a field $\mathbf{k}$ with a faithful representation V of dimension n, the ring $CH^\bullet BG$ is generated by elements of degree at most $n(n-1)/2$ if $n \geqslant 3$, or of degree at most n if $n \leqslant 2$.*

3.5 Chow ring of a quotient stack

The Chow groups of a quotient stack $[X/G]$ are defined to be the equivariant Chow groups of X:

$$CH_\bullet[X/G] := CH_\bullet^G(X).$$

Example 3.15. $[*/G]$ is represented by BG, and

$$CH_\bullet[*/G] = CH_\bullet^G(*) = CH_\bullet(BG).$$

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