Flavours of Equivariant K-Theory

'Michael' Zeng, Ruofan

May 21, 2024

Contents

1 Equivariant Topological K-Theory 1
  1.1 Topological K-Theory 1
  1.2 Equivariant Topological K-Theory 2

2 Equivariant Algebraic K-Theory 5
  2.1 Algebraic K-Theory of Rings 5
  2.2 Algebraic K-Theory for Categories 6
  2.3 Equivariant K-Theory of Schemes 7

3 Equivariant Algebraic K-Theory of G-Rings 8
  3.1 Modules over G-Rings 8
  3.2 Homotopy Fixed Points of a G-Category 8
  3.3 Pseudo Equivariant Functors 10
  3.4 Twisted Group Rings 11
  3.5 Equivariant K-Theory of G-Rings 12

1 Equivariant Topological K-Theory

1.1 Topological K-Theory

Topological K-theory was first formulated by Atiyah. Hoping to derive an analogue of Grothendieck-Riemann-Roch in the smooth setting, K-theory was developed for the definition of topological index in the renowned Atiyah-Singer index theorem. One of the hallmarks of K-theory was Adams' solution to the Hopf invariant one problem, which implies the parallelizability of spheres and existence of division algebras over the reals.

For the ease of exposition, we fix our base field $\mathbf{k} = C$ unless otherwise specified.

Definition 1.1. Let X be a compact Hausdorff space and $\mathrm{Vect}_{\mathbb{C}}(X)$ the set of isomorphism classes of finite rank complex vector bundles over X. The **K-theory ring** $K^0(X)$ is defined to be the Grothendieck group of $\mathrm{Vect}_{\mathbb{C}}(X)$, i.e. the stable isomorphism classes of formal differences of elements in $\mathrm{Vect}_{\mathbb{C}}(X)$, i.e.

$$\{[E] - [F] \mid [E], [F] \in \mathrm{Vect}_{\mathbb{C}}(X)\} / \sim,$$

where

$$[E_1] - [F_1] \sim [E_2] - [F_2] \text{ if there exists } [W] \text{ such that } E_1 \oplus F_2 \oplus W \cong E_2 \oplus F_1 \oplus W.$$

Equivalently,

$$[E] = [E'] + [E''] \text{ if there exists a short exact sequence } 0 \to E' \to E \to E'' \to 0.$$

Addition and multiplication of $K^0(X)$ are given by the direct sum and tensor product.

Definition 1.2. For X with a base point, the **reduced K-theory** $\widetilde{K}^0(X)$ is defined to be the kernel of the map $K^0(X) \to K^0(\mathrm{pt})$ given by restricting vector bundles over X to the base point.

Proposition 1.3.

$$\tilde{K}^0(X_+) \cong K^0(X).$$

Definition 1.4. The negative degree piece $K^{-n}(X)$ is defined to be $K^0(\mathbb{R}^n \times X)$.

While K^0 is itself a commutative ring, the negative degree pieces fit together with the former as a graded commutative ring. To see this, consider the external product induced by the projection maps

$$\begin{tikzcd} & X \times Y \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\ X & & Y. \end{tikzcd}$$

The external product $K^{-n}(X) \otimes K^{-m}(Y) \to K^{-n-m}(X \times Y)$ is defined as

$$[E] \otimes [F] \mapsto [\pi_X^* E \otimes \pi_Y^* F].$$

Then, for the internal product, we set $Y = X$ and pull back along the diagonal:

Proposition 1.5. *The multiplication*

$$K^{-n}(X) \otimes K^{-m}(X) \to K^{-n-m}(X \times X) \xrightarrow{\Delta^*} K^{-n-m}(X)$$

makes $\bigoplus_{n=0}^{\infty} K^{-n}(X)$ *into a commutative graded ring.*

Bott periodicity allows us to either extend the definition to all $\mathbb{Z}$-degrees or change the grading into $\mathbb{Z}/2$.

Theorem 1.6 (Bott periodicity). *There is an isomorphism* $K^{-n}(X) \to K^{-n-2}(X)$, *in the form of multiplication by the* ***Bott class****.*

A similar periodicity holds for real K-theory, where it becomes 8-periodic.

Theorem 1.7. *Topological K-theory is a generalized cohomology theory.*

1.2 Equivariant Topological K-Theory

Like Borel equivariant cohomology for G-spaces, topological K-theory admits an equivariant version on the category of G-spaces. This was introduced by Segal in his PhD thesis. Fok 2023

Definition 1.8. Let G be a compact Lie group acting on a locally compact Hausdorff space X. An G-equivariant vector bundle on X is a fiber bundle $V \to E \to X$ where

(a) V is a complex G-representation.

(b) The structure map $\pi : E \to X$ is G-equivariant.

Definition 1.9. Let $\mathrm{Vect}_{\mathbb{C}}^{G}(X)$ be the set of isomorphism classes of finite rank complex G-vector bundles over X. For X compact, we define the **equivariant K-theory ring** $K_G^0(X)$ to be the Grothendieck group of $\mathrm{Vect}_{\mathbb{C}}^{G}(X)$, with addition and multiplication given by $\oplus$ and $\otimes$.

Reduced equivariant K-theory is defined as the kernel of $K_G^0(X) \to K_G^0(\mathrm{pt})$ *whenever X has a G-fixed point*. Same as before, for non-compact X, define $K_G^0(X)$ to be $\widetilde{K}_G^0(X_+)$ where G fixes the point of compactification. Define the negative degree piece $K_G^{-n}(X)$ to be $K_G^0(\mathbb{R}^n \times X)$, where $\mathbb{R}^n$ is the trivial representation. Define the product structure by pulling back the external product along the diagonal. A similar Bott periodicity holds:

Theorem 1.10 (Segal '68)**.** *$K_G^{-q}(\mathrm{X})$ is naturally isomorphic to $K_G^{-q-2}(\mathrm{X})$, the map being multiplication by a certain element of $K_G^{-2}(\mathrm{pt})$ (**equivariant Bott class**).*

Example 1.11. A G-equivariant vector bundle over a point is simply a complex G-representation. Therefore,

$$K_G^0(\mathrm{pt}) \cong R_{\mathbb{C}}(G)$$

where the latter is the representation ring of G.

Example 1.12. If H is a Lie subgroup of G, then $K_G^\bullet(G/H) \cong K_H^\bullet(\mathrm{pt}) \cong R_{\mathbb{C}}(H)$. This is because a G-equivariant vector bundle over G/H is determined by the restricted H-action on the fiber over the identity coset $e_G H$.

Example 1.13. For a G-space X, a Lie group homomorphism $\varphi : H \to G$ induces an H-action on X, so it induces a homomorphism $K_G^\bullet(X) \to K_H^\bullet(X)$.

The following properties of equivariant K-theory are in parallel with Borel cohomology.

Example 1.14. If G acts on X trivially, then $K_G^\bullet(X)$ becomes isomorphic to $R_{\mathbb{C}}(G) \otimes K^\bullet(X)$.

Example 1.15. If G acts freely on X, then there is an isomorphism of commutative monoids $\mathrm{Vect}_{\mathbb{C}}^{G}(X) \cong \mathrm{Vect}_{\mathbb{C}}(X/G)$, so

$$K_G^\bullet(X) \cong K^\bullet(X/G).$$

Proposition 1.16. *The constant map $X \to \mathrm{pt}$ endows $K_G^\bullet(X)$ the structure of an $R_\mathbb{C}(G)$-algebra.*

The great feature of localization also holds for equivariant K-theory.

Theorem 1.17 (**Segal localization**, toric version)**.** *Let T be a compact torus acting on a locally compact Hausdorff space X. Then the restriction map*

$$i^* : K_T^*(X) \to K_T^*\left(X^T\right) \cong R_\mathbb{C}(T) \otimes K^*\left(X^T\right)$$

becomes an isomorphism after localizing the coefficient ring $R_\mathbb{C}(T) \cong \mathbb{Z}[x_1^{\pm 1}, \ldots, x_k^{\pm 1}]$ at the zero ideal.

Theorem 1.18 (**Segal localization**, general version)**.** *Let $\mathfrak{p}$ be a prime ideal of $R_\mathbb{C}(G)$, A an abelian subgroup supporting $\mathfrak{p}$, and X a locally compact Hausdorff G-space. Then the restriction map*

$$i^* : K_G^*(X) \to K_G^*\left(G \cdot X^A\right)$$

becomes an isomorphism after localizing the coefficient ring $R_\mathbb{C}(G)$ at $\mathfrak{p}$.

The toric localization theorem implies a K-theoretic version of the fixed point formula, called **Atiyah-Segal localization formula**.

2 Equivariant Algebraic K-Theory

2.1 Algebraic K-Theory of Rings

Constructions in topological K-theory generalize in many ways to algebraic settings. The Serre-Swan theorem establishes a connection between vector bundles and projective modules. Therefore, it is natural to consider the Grothendieck group construction on projective modules. Weibel 2013

Definition 2.1. For R associative unital, let $Proj(R)$ denote the isomorphism classes of finitely generated projective R-modules. The zeroth algebraic K-group of the ring R is

$$K_0R := Gr(Proj(R)).$$

Define the reduced zeroth K-group by modding out the image of $\mathbb{Z} \to K_0R, n \mapsto [R^{\oplus n}]$.

Unlike the topological situation, it is not as clear how to extend this definition to higher graded pieces. Classically, K_1 and K_2 were defined by allusion to group homology.

Definition 2.2. Let $[-,-]$ denote taking commutators. The first algebraic K-group of the ring R is defined as

$$K_1R := GL(R)/[GL(R), GL(R)].$$

Let e_{ij} be the elementary matrix with 1s on the diagonal and at position (i, j). Define the subgroup of $GL(R)$,

$$E(R) := \langle re_{ij} \mid r \in R, i, j \in \mathbb{N} \rangle.$$

Then, we have equivalently,

$$K_1R := GL(R)/E(R).$$

Definition 2.3. The **Steinberg group** $St(R)$ is defined via the following generators

$$\{x_{ij}^r \mid r \in R, i \neq j\}$$

and relations

$$\begin{cases} x_{ij}^r x_{ij}^s = x_{ij}^{r+s} \\ \left[x_{ij}^r, x_{j\ell}^s\right] = x_{i\ell}^{rs} & \text{if } i \neq \ell, \\ \left[x_{ij}^r, x_{k\ell}^s\right] = 1 & \text{if } i \neq \ell \text{ and } j \neq k \end{cases}$$

Let $\varphi : St(R) \to E(R)$ be the surjective group homomorphism sending $x_{ij}^r \mapsto re_{ij}$. Then, we define

$$K_2R := \ker \varphi.$$

Theorem 2.4 (Kervaire, Steinberg)**.** *The surjection $St(R) \to E(R)$ is the universal central extension of $E(R)$. As a consequence,*

$$K_2(R) \cong H_2(E(R), \mathbb{Z})$$

Extending these definitions following group homology becomes extremely cumbersome after K_2. Quillen in the '70s came up with the brilliant idea to use the **classifying space functor** to transform this into a problem in homotopy theory.

Definition 2.5. (Quillen) The **higher algebraic K-groups** are defined to be the homotopy groups of $BGL(R)^+$.

$$K_n(R) := \pi_n BGL(R)^+$$

The $^+$ stands for the **plus construction**. The key observation is that simply taking π_n of the space $BGL(R)$ results in $\pi_1 BGL(R) = GL(R)$, but we really want the abelianization $GL(R)/E(R)$. The plus construction modifies cells in the classifying space in order to achieve that.

2.2 Algebraic K-Theory for Categories

Using these ideas, Quillens extended the definition of K-groups to any exact category.

Definition 2.6. Let $\mathcal{C}$ be an exact category. The category $Q\mathcal{C}$ has object $ob(\mathcal{C})$ and morphisms

$$Hom_{Q\mathcal{C}}(X, Y) := \{X \leftarrow Z \to Y\}$$

where the first arrow is an admissible epi and the second admissible mono.

Let Ω denote the loop space functor. We can apply homotopy theoretic constructions again and define the K-groups:

Definition 2.7. The K-groups of the exact category $\mathcal{C}$ are

$$K_i(\mathcal{C}) := \pi_{i+1}(\Omega BQ\mathcal{C})$$

2.3 Equivariant K-Theory of Schemes

For a scheme X, we may replace $\mathrm{Vect}(X)$ with $\mathrm{Coh}(X)$, the abelian category of coherent sheaves over X, and carry out the same Grothendieck group constructions.

Definition 2.8. Let $\mathrm{Coh}^G(X)$ be the category of equivariant coherent sheaves on an algebraic scheme X with action of a linear algebraic group G. The **equivariant algebraic K-groups** are defined as

$$K_i^G(X) = \pi_{i+1}\left(BQ\,\mathrm{Coh}^G(X)\right).$$

3 Equivariant Algebraic K-Theory of G-Rings

The previous construction is the algebraic analogue for K-theory of equivariant vector bundles, so what we are missing is an equivariant theory for rings with a group action. These are ubiquitous: $\mathbf{k}[x_1, \ldots, x_n]$ has a natural $\mathcal{S}_n$-action, $\mathbb{C}$ has a C_2-action by conjugation, and so on. The goal is to use the machinery at hand and construct a genuine G-spectrum from a G-ring R. Merling 2016

3.1 Modules over G-Rings

Definition 3.1. A G-ring is a ring R with a left action $G \to \mathrm{Aut}(R)$. We write $g(r) = {}^g r$ for the automorphism $g : R \to R$ determined by $g \in G$. Then ${}^{gh}r = g(h(r)) = {}^g({}^h r)$.

If R is a G-ring, then $\mathrm{Proj}(R)$ automatically has a G-action, where gM is defined by twisting the scalar multiplication on R by g.

Definition 3.2. For a R-module M – equivalently, $\varphi : R \to \mathrm{End}_{\mathbf{Ab}} M$, define $g \cdot M$ as

$$g \cdot \varphi : R \xrightarrow{g} R \to \mathrm{End}_{\mathbf{Ab}} M.$$

In other words, $g \cdot M$ has the same underlying abelian group as M, but the scalar multiplication becomes

$$r \cdot_{gM} m := {}^g r \cdot_M m.$$

Similarly, we have for free a G-action on the category of modules over a G-ring spectrum R.

However, by applying the nonequivariant constructions to this category with G-action, we obtain just a spectrum with G-action, and not a genuine G-spectrum - the K-theory G-space we obtain has deloopings with respect to all spheres S^n with trivial G-action, but it does not deloop with respect to representation spheres S^V. To turn these into genuine G-categories, Merling introduced homotopy fixed points and pseudo equivariant functors for categories with a G-action.

3.2 Homotopy Fixed Points of a G-Category

Definition 3.3. A G**-category** is a functor $G \to \mathbf{Cat}$. Explicitly, the data of such a functor is a category $\mathscr{C}$, and for each $g \in G$, an endofunctor $(g\cdot) : \mathscr{C} \to \mathscr{C}$ such that $(e\cdot) = \mathrm{id}_{\mathscr{C}}$ and $(g\cdot) \circ (h\cdot) = (gh)\cdot$.

Denote the category of G-categories and G-equivariant functors by $G\mathbf{Cat}$.

Definition 3.4. For subgroups $H \subseteq G$, we define the H**-fixed point category** $\mathscr{C}^H$ of a G-category $\mathscr{C}$ as the subcategory with objects those $C \in \mathscr{C}$ such that $hC = C$ and morphisms those $f \in \mathscr{C}$ such that $hf = f$ for all $h \in H$. This definition coincides with the categorical definition as $\lim_H \mathscr{C}$.

Proposition 3.5. *The classifying space functor B : **Cat** $\to$ **Top** commutes with fixed points, namely*

$$B(\mathscr{C}^H) = (B\mathscr{C})^H.$$

Definition 3.6. A functor between G-categories $F : \mathscr{C} \to \mathscr{D}$ is a **weak G-equivalence** if it induces a weak G-equivalence on classifying spaces $BF : B\mathscr{C} \to B\mathscr{D}$.

Remark 3.7. Similar to **Cat**, G**Cat** has the structure of a 2-category whose 0, 1, and 2-morphisms are G-categories, G-equivariant functors, and G-natural transformations.

Recall, the homotopy fixed points for a G-space X is defined to be $\mathrm{Maps}(EG, X)^G$. The following will be our model for EG.

Definition 3.8. For a topological group G, define $\widetilde{G}$ to be the topological G-groupoid with object space G and morphism space $G \times G$.

Remark 3.9. The classifying space $B\widetilde{G}$ is G-equivalent to the universal principal G-bundle EG since $\widetilde{G}$ is a contractible category (every object is initial and terminal) and it has a free G-action.

Definition 3.10. The **homotopy fixed points** of a G-category $\mathscr{C}$, denoted by $\mathscr{C}^{hG}$, are defined as $\mathbf{Cat}(\widetilde{G}, \mathscr{C})^G$, namely the G-equivariant functors $\widetilde{G} \to \mathscr{C}$ and the G-natural transformations between these.

We have the following explicit description of $\mathscr{C}^{hG}$:

Proposition 3.11. *The objects of the homotopy fixed point category $\mathscr{C}^{hG}$ are pairs (C, f) where C is an object of $\mathscr{C}$ and $f : G \to \mathrm{Mor}(\mathscr{C})$ is a map from G to morphisms of $\mathscr{C}$ such that $f(g) : C \to g \cdot C$ and f satisfies the condition $f(e) = id_C$ and the cocycle condition*

$$f(gh) = f(g)(g \cdot f(h)).$$

A morphism $(C, f) \to (C', f')$ is given by a morphism $\alpha : C \to C'$ in $\mathscr{C}$ such that the following diagram commutes for any $g \in G$:

$$\begin{tikzcd} C \arrow[r, "f(g)"] \arrow[d, "\alpha"'] & gC \arrow[d, "g\alpha"] \\ C' \arrow[r, "f'(g)"] & gC' \end{tikzcd}$$

In the special case where $\mathscr{C} = \Pi$ is a group, the above cocycle condition takes on a familiar meaning:

Theorem 3.12. *Suppose Π is a group with G-action. The homotopy fixed point category Π^{hG} is equivalent to the crossed functor category $\textbf{Cat}_\times(G, \Pi)$ whose objects are **crossed homomorphisms** $G \to \Pi$ and whose morphisms $\sigma : \alpha \to \beta$ are the elements $\sigma \in \Pi$ such that*

$$\beta(g)(g \cdot \sigma) = \sigma\alpha(g).$$

3.3 Pseudo Equivariant Functors

Definition 3.13. A **pseudo equivariant functor** between G-categories $\mathscr{C}$ and $\mathscr{D}$ is a functor $\Theta : \mathscr{C} \to \mathscr{D}$, together with natural isomorphisms of functors θ_g for all $g \in G$

$$\begin{tikzcd} \mathscr{C} \arrow[r, "g"] \arrow[d, "\Theta"'] & \mathscr{C} \arrow[d, "\Theta"] \arrow[dl, Rightarrow, "\theta_g"] \\ \mathscr{D} \arrow[r, "g"'] & \mathscr{D} \end{tikzcd}$$

such that $\theta_e = \mathrm{id}$ and for $g, h \in G$ we have an equality of natural transformations,

$$
\begin{tikzcd}
\mathscr{C} \arrow[r, "g"] \arrow[d, "\Theta"'] & \mathscr{C} \arrow[r, "h"] \arrow[d, dashed, "\Theta"] \arrow[dl, phantom, "\swarrow\theta_g"] & \mathscr{C} \arrow[d, "\Theta"] \arrow[dl, phantom, "\swarrow\theta_h"] \\
\mathscr{D} \arrow[r, "g"'] & \mathscr{D} \arrow[r, "h"'] & \mathscr{D}
\end{tikzcd}
=
\begin{tikzcd}
\mathscr{C} \arrow[r, "gh"] \arrow[d, "\Theta"'] & \mathscr{C} \arrow[d, "\Theta"] \arrow[dl, phantom, "\swarrow\theta_{gh}"] \\
\mathscr{D} \arrow[r, "gh"'] & \mathscr{D}
\end{tikzcd}
$$

Proposition 3.14. *A pseudo equivariant functor $\Theta : \mathscr{C} \to \mathscr{D}$ naturally induces an equivariant functor*

$$\tilde{\Theta} : \boldsymbol{Cat}(\tilde{G}, \mathscr{C}) \to \boldsymbol{Cat}(\tilde{G}, \mathscr{D}).$$

Corollary 3.15. *A pseudo equivariant functor $\Theta : \mathscr{C} \to \mathscr{D}$, induces functors $\tilde{\Theta}^H : \mathscr{C}^{hH} \to \mathscr{D}^{hH}$ on homotopy fixed points for all $H \subseteq G$.*

Corollary 3.16 (Homotopy invariance of homotopy fixed points). *A pseudo equivariant functor $\Theta : \mathscr{C} \to \mathscr{D}$ which is a nonequivariant equivalence induces equivalences of homotopy fixed points*

$$\mathscr{C}^{hH} \to \mathscr{D}^{hH}$$

for all $H \subseteq G$.

3.4 Twisted Group Rings

In the non-equivariant setting, the set of G-representations over a ring R and the set of $R[G]$-modules are isomorphic as abelian monoids. We extend this to equivariantly.

Suppose that R is a commutative G-ring with action given by $\theta : G \to \mathrm{Aut}(R)$. Observe that R is an R^G-algebra, where R^G is the subring of G-invariants. We can reinterpret θ as a group homomorphism $\theta : G \to \mathrm{End}_{R^G} R$, and ask the question of when we can extend this to a ring map. More precisely, we seek to extend θ R^G-linearly from G to the whole of $R[G]$. This leads to the definition of a twisted group ring.

Definition 3.17. As an R-module, the **twisted group ring** $R_G[G]$ is the same as the group ring $R[G]$, which is the case when G acts trivially on R. We define the product on $R_G[G]$ by R^G-linear (not R-linear) extension of the relation

$$(rg)(sh) = r^g sgh$$

for $r, s \in R$ and $g, h \in G$.

Much like the case for G-linear actions, we have the bijective correspondence between $R_G[G]$**-modules** and **semilinear** G**-representations**, i.e., $g(rm) = {}^{g}r(gm)$ for $m \in M$. If the action of G on R is trivial, then we recover the usual correspondence: an $R[G]$-module is a left R-module M with linear G-action, namely, $g(rm) = r(gm)$. From this point of view an $R_G[G]$-linear map of $R_G[G]$-modules $f : M \to N$ is a map of R-modules, which commutes with the G-action.

We now state a few results that identifies module categories over $R_G[G]$ with homotopy fixed points.

Proposition 3.18. *If G is finite and $|G|^{-1} \in R$, then an $R_G[G]$-module is projective if and only if it is projective as an R-module.*

Proposition 3.19. *The homotopy fixed point category* $\mathrm{Mod}(R)^{hG}$ *is equivalent to the category* $\mathrm{Mod}\,(R_G[G])$.

Proposition 3.20. *Suppose G is finite and $|G|^{-1} \in R$. The homotopy fixed point category* $\mathrm{Proj}(R)^{hG}$ *is equivalent to the category* $\mathrm{Proj}\,(R_G[G])$.

3.5 Equivariant K-Theory of G-Rings

In order to get a genuine G-spectrum, we need to devise an equivariant delooping machine.

Definition 3.21. A **Hopf** G**-space** is an H-space with equivariant multiplication map and for which multiplying by the identity element is G-homotopic to the identity map such as, for example, ΩX for a G-space X.

A G-map $X \to Y$ of homotopy associative and commutative Hopf G-spaces is an **equivariant group completion** if the fixed point maps $X^H \to Y^H$ are group completions for all $H \subseteq G$

We extend the following non-equivariant construction by Quillen to show the existence of such equivariant completions.

Definition 3.22. Let S be a symmetric monoidal category. The category $S^{-1}S$ has objects pairs (m, n) of objects in S. A morphism $(m, n) \to (p, q)$ in $S^{-1}S$ is an equivalence class of triples

$$(r, r \oplus m \xrightarrow{f} p, r \oplus n \xrightarrow{g} q)$$

where two triple are equivalent if there is an isomorphism of the first entries that makes the relevant diagrams commute. Composition for a pair of morphisms is defined as

$$(r, r \oplus m \xrightarrow{f} p, r \oplus n \xrightarrow{g} q) \circ (s, s \oplus p \xrightarrow{\phi} u, s \oplus q \xrightarrow{\psi} v)$$
$$= (s \oplus r, s \oplus r \oplus m \xrightarrow{\phi \circ (s \oplus f)} u, s \oplus r \oplus n \xrightarrow{\psi \circ (s \oplus g)} v).$$

Theorem 3.23 (Quillen '73)**.** *Let S be a symmetric monoidal groupoid such that translations are faithful. i.e.,*

$$\mathrm{Aut}(s) \to \mathrm{Aut}(s \oplus t)$$

is injective for all $s, t \in S$. Then the map $BS \to BS^{-1}S$ is a group completion.

Theorem 3.24 (Merling '16)**.** *Let S be a symmetric monoidal G-groupoid such that translations are faithful. Then the map $BS \to BS^{-1}S$ is an equivariant group completion.*

Finally, the payoff.

Definition 3.25. The **equivariant algebraic K-theory space** of a G-ring R is the G-space $K_G(R) = B\left(S^{-1}S\right)$, where S is the symmetric monoidal G-category **Cat**$(\widetilde{G}$, $\mathrm{Proj}(R))$.

For H a subgroup of G, the **equivariant algebraic K-theory groups** are given by

$$K_i^H(R) = \pi_i^H\left(K_G(R)\right).$$

These definitions naturally extends to a genuine $\Omega - G$-spectrum.
Below are a few of the consequences of this definition.

Theorem 3.26. *The assignment $R \to K_G(R)$ is functorial and factors through equivariant Morita equivalences.*

Theorem 3.27. *For the topological rings $\mathbb{C}$ and $\mathbb{R}$ with trivial G-action for any finite group G,*

$$\mathbf{K_G}(C) \simeq ku_G \text{ and } \mathbf{K_G}(R) \simeq ko_G,$$

where ku_G and ko_G are connective versions of equivariant topological K-theory.

Theorem 3.28. *For the topological ring $\mathbb{C}$ with C_2 conjugation action*

$$\mathbf{K}_{\mathbf{C}_2}(C) \simeq kr,$$

where kr is a connective version of Atiyah's Real K-theory.

Theorem 3.29. *For a Galois extension of rings $R \to S$ with Galois group G,*

$$\mathbf{K}_{\mathbf{G}}(S)^G \simeq \mathbf{K}(R).$$

Michael's speculation: connection with equivariant motivic cohomology?

References

Fok, Chi-Kwong (2023). "A stroll in equivariant *K*-theory". In: arXiv: 2306.06951 [math.KT].

Merling, Mona (2016). "Equivariant algebraic K-theory of G-rings". In: arXiv: 1505.07562 [math.AT].

Weibel, C.A. (2013). The K-book: An Introduction to Algebraic K-theory. Graduate Studies in Mathematics. American Mathematical Society. ISBN: 9781470409432. URL: https://books.google.com/books?id=YrnZjwEACAAJ.