The K_0-Ring of Spanning Line Configurations

Michael R. Zeng

Main Result

Theorem

There are isomorphisms

$$K_0(X_{n,k}) \cong H^\bullet(X_{n,k}) \cong \frac{\mathbb{Z}[x_1, x_2, \ldots, x_n]}{(x_1^k, x_2^k, \ldots, x_n^k, e_{n-k+1}, e_{n-k+2}, \ldots, e_n)}.$$

Spanning Line Configurations

Cell Decomposition & Fubini Words

Cohomology

A Tale of Two Grothendiecks

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Spanning Line Configurations

A *line configuration* of length n in $\mathbb{C}^k$ is an n-tuple $\ell_\bullet = (\ell_1, \ell_2, \ldots, \ell_n) \in (\mathbb{C}^k)^n$ where each ℓ_i is a 1-dimensional linear subspace (a *line*) in $\mathbb{C}^k$.

$$\ell_\bullet \quad \rightsquigarrow \quad m_\ell = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{k1} & x_{n2} & \cdots & x_{kn} \end{pmatrix} \quad \rightsquigarrow \quad \text{point of } (\mathbb{P}^{k-1})^n$$

A *spanning line configuration* is a line configuration whose columns span $\mathbb{C}^k$:

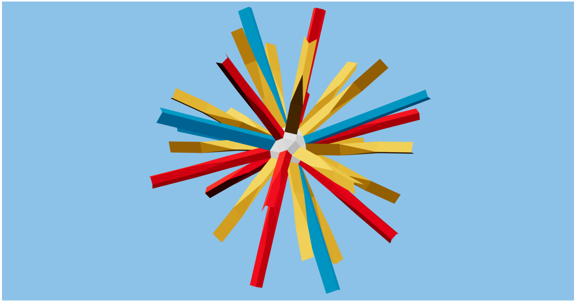
$$\ell_1 + \ell_2 + \cdots + \ell_n = \mathbb{C}^k \quad \rightsquigarrow \quad m_\ell \text{ full rank.}$$

Spanning Line Configurations

The *space of spanning line configurations* is

$$X_{n,k} := \{\ell_\bullet \mid \ell_1 + \cdots + \ell_n = \mathbb{C}^k\}.$$

The space $X_{n,k}$ is an open set of $(\mathbb{P}^{k-1})^n$ and a smooth quasiprojective variety.



Created using vZome [Vor]

Generalized Coinvariant Algebra

A *complete flag* in $\mathbb{C}^n$ is a chain of nested subspaces $0 = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_n = \mathbb{C}^n$. The variety of complete flags is denoted $Fl_n(\mathbb{C})$.

Borel presentation.

$$H^\bullet(Fl_n(\mathbb{C})) \cong \frac{\mathbb{Z}[x_1, \ldots, x_n]}{(e_1, \ldots, e_n)}.$$

Fix $k \leq n$. The *generalized coinvariant algebra* is

$$R_{n,k} := \frac{\mathbb{Z}[x_1, \ldots, x_n]}{(x_1^k, \ldots, x_n^k, e_{n-k+1}, \ldots, e_n)}.$$

Cohomology

[PR19, Theorem 5.12]

There is an isomorphism

$$H^\bullet(X_{n,k}) \cong R_{n,k}.$$

Special case

$k = n$. $X_{n,n}$ is homotopy equivalent to Fl_n, and $R_{n,n}$ is the classical coinvariant algebra.

Importance

$R_{n,k}$ comes up in symmetric function theory and orbit harmonics [HRS18] [OR22].

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Cell Decomposition

The variety $X_{n,k}$ is *cellular*:

$$X_{n,k} = \bigsqcup_{w \in \text{Fubini}(n,k)} C_w$$

Each *Pawlowski-Rhoades* cell C_w is isomorphic to $\mathbb{C}^{\mathrm{inv}(w)}$.

Closure relations $\overline{C}_w = \bigcup_{v \leq w} C_v$.

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$$C_{2331231} = \left\{ U \cdot \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & \star & \star & 0 & 1 & \star & \star \\ 0 & 1 & 1 & 0 & 0 & 1 & \star \end{pmatrix} \middle| \star \in \mathbb{C} \right\}.$$

U: lower triangular matrices with 1's on diagonal

Fubini Words

A *word* of length n with alphabet $[k]$ is an element of $\mathrm{Words}(n,k) := [k]^n$.

A *Fubini word* is a word in which every letter appears.

$$\begin{aligned} \mathrm{Fubini}(n,k) \quad &\rightsquigarrow \quad \text{surjective functions } [n] \twoheadrightarrow [k] \\ &\rightsquigarrow \quad \text{ordered set partitions of } [n] \text{ with } k \text{ parts.} \end{aligned}$$

$$\underset{1\,2\,3\,4\,5\,6\,7}{2331231} \quad \rightsquigarrow \quad \{47 \mid 15 \mid 236\}.$$

Fubini Words

Variety $X_{n,k}$	Set $\mathrm{Fubini}(n,k)$
PR cells	Fubini words
Cell containment	Fubini-Bruhat order (*medium roast*) [BS24]
Cohomology classes	Schubert polynomials of words

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Cohomology: Proof

$$X_{n,k} = \left(\mathbb{P}^{k-1}\right)^n - \bigcup_{w \in \mathrm{Words}(n,k) - \mathrm{Fubini}(n,k)} \overline{C}_w \quad .$$

Localization Theorem [PR19, Theorem 4.5]

- X: smooth complex cellular algebraic variety.
- $Z = \bigcup_{j \in J} \overline{C}_j \subseteq X$: union of cell closures.
- Ideal $I(Z) := ([\overline{C}_j] \mid j \in J)$.

Then,

$$H^\bullet(X - Z) \cong H^\bullet(X)/I(Z).$$

Cohomology: Proof

$$X_{n,k} = \left(\mathbb{P}^{k-1}\right)^n - \bigcup_{w \in \mathrm{Words}(n,k) - \mathrm{Fubini}(n,k)} \overline{C}_w$$

$$\implies H^\bullet(X_{n,k}) = \frac{H^\bullet((\mathbb{P}^{k-1})^n)}{I(\overline{C}_w \mid w \text{ not Fubini })}$$

(Fulton's degeneracy loci formula) [Ful91]

$$= \frac{\mathbb{Z}[x_1, \ldots, x_n]/(x_1^k, \ldots, x_n^k)}{(\mathfrak{S}_{23..n1}, \ldots, \mathfrak{S}_{1..\hat{i}..ni}, \ldots, \mathfrak{S}_{1..\hat{k}..nk})}$$

$$= \frac{\mathbb{Z}[x_1, \ldots, x_n]/(x_1^k, \ldots, x_n^k)}{(e_{n-k+1}, e_{n-k+2}, \ldots, e_n)}$$

$$= \frac{\mathbb{Z}[x_1, x_2, \ldots, x_n]}{(x_1^k, x_2^k, \ldots, x_n^k, e_{n-k+1}, e_{n-k+2}, \ldots, e_n)}.$$

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The Grothendieck Group

The *Grothendieck group* $K_0(X)$ of a variety X is the abelian group

$$\frac{\mathbb{Z} \cdot \textsc{Iso}\, \underline{\mathrm{VB}}_X}{[B] = [A] + [C] \;\text{ if }\; 0 \to A \to B \to C \to 0}$$

It 'retains more information than cohomology' in the sense that the associated graded of K_0 surjects onto Chow groups.

$$\mathrm{gr}_\gamma\, K_0(X) \twoheadrightarrow CH^\bullet(X).$$

Ring Structure

If X is smooth, then direct sums $\oplus$ and tensor products $\otimes$ of bundles make $K_0(X)$ a commutative ring.

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Examples

- $K_0(\mathbb{P}^{k-1}) \cong \mathbb{Z}[x]/(x^k)$.
- $K_0((\mathbb{P}^{k-1})^n) \cong \mathbb{Z}[x_1, \ldots, x_n]/(x_1^k, \ldots, x_n^k)$.
- [Len05, p. 121]

$$K_0(Fl_n) \cong \frac{\mathbb{Z}[x_1, \ldots, x_n]}{(e_1, \ldots, e_n)}.$$

Grothendieck Polynomials

Isobaric divided difference operator

$$\pi_i f(x) = \frac{(1 - x_{i+1})\, f(x) - (1 - x_i)\, s_i f(x)}{x_i - x_{i+1}}$$

The *Grothendieck polynomial* [Las90] of a permutation $\omega \in \mathcal{S}_n$ is

$$\mathfrak{G}_\omega = \begin{cases} x_1^{n-1} x_2^{n-2} \cdots x_{n-1} & \text{if } \omega = \omega_0 \\ \pi_i \mathfrak{G}_{\omega s_i}(x) & \text{if } \omega(i) < \omega(i+1). \end{cases}$$

Facts

- Lowest degree component of $\mathfrak{G}_\omega$ is $\mathfrak{S}_\omega$.
- $\mathfrak{G}_\omega$ represents the Schubert class $[X_\omega]$ in $K_0(Fl_n)$.

Double Grothendieck Polynomials

β-Demazure operator

$$\pi_i := \frac{(1+\beta x_{i+1})f - (1+\beta x_i)\, s_i(f)}{x_i - x_{i+1}}.$$

The *Double Grothendieck polynomial* [FL94] of a permutation $\omega \in \mathcal{S}_n$ is

$$\mathfrak{G}_\omega(\overline{\mathbf{x}} \mid \overline{\mathbf{b}}) = \begin{cases} \prod_{i+j\leq n} \left(x_i + b_j + \beta x_i b_j\right) & \text{if } \omega = \omega_0 \\ \pi_i \mathfrak{G}_\omega(\overline{\mathbf{x}} \mid \overline{\mathbf{b}}) & \text{if } \omega(i) < \omega(i+1). \end{cases}$$

Fact

Double Grothendieck polynomials are closely related to *bumpless pipedreams* [Wei21].

Degeneracy Loci Formula for K_0

Fulton-Lascoux Degeneracy Loci Formula for K_0
[FL94, Theorem 4]

Classes of degeneracy loci in K_0 are given by double Grothendieck polynomials in K-theoretic first Chern classes.

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Towards K_0 of $X_{n,k}$

$$X_{n,k} = \left(\mathbb{P}^{k-1}\right)^n - \bigcup_{w \in \mathrm{Words}(n,k) - \mathrm{Fubini}(n,k)} \overline{C}_w .$$

A Localization Theorem for K_0 [Z.]

- X: smooth complex cellular algebraic variety, number of cells finite.
- $Z = \bigcup_{j \in J} \overline{C}_j \subseteq X$: union of cell closures.
- Ideal $I(Z) := ([\mathcal{O}_{\overline{C}_j}] \mid j \in J)$.

Then,

$$K_0(X - Z) \cong K_0(X)/I(Z).$$

TOWARDS K_0 OF $X_{n,k}$

$$X_{n,k} = \left(\mathbb{P}^{k-1}\right)^n - \bigcup_{w \in \mathrm{Words}(n,k) - \mathrm{Fubini}(n,k)} \overline{C}_w \qquad .$$

$$\implies K_0(X_{n,k}) = \frac{K_0((\mathbb{P}^{k-1})^n)}{I(\overline{C}_w \mid w \text{ not Fubini })}$$

(F& L degeneracy loci formula) [FL94] $\cong \dfrac{\mathbb{Z}[x_1, \ldots, x_n]/(x_1^k, \ldots, x_n^k)}{(\mathfrak{G}_{23..n1}, \ldots, \mathfrak{G}_{1..\hat{i}..ni}, \ldots, \mathfrak{G}_{1..\hat{k}..nk})}$

K_0-Elementary Symmetrics

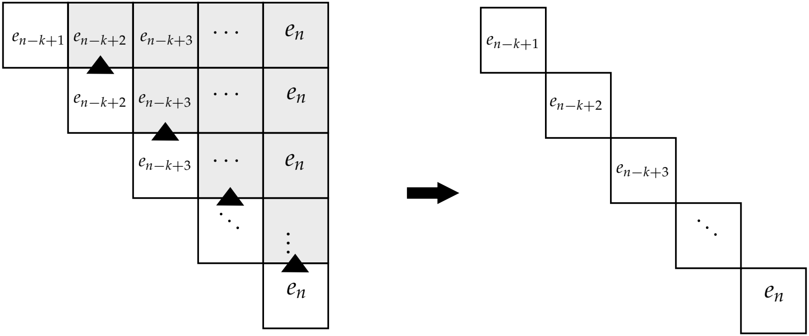
K-class of Grassmannian permutations [Len00, Theorem 2.2]

$\omega_i := 12\ldots\hat{i}\ldots n(n+1)i \in S_{n+1}$ Grassmannian with shape of vertical column.

$$\begin{aligned}\mathfrak{G}_{\omega_i} &= e_{n+1-i} - g_{n+2-i}e_{n+2-i} + \cdots + (-1)^{i+1}g_n e_n \\ &= e_{n+1-i} + \sum_{n+1-i<j\leq n} (-1)^{i+j-n-1} g_j e_j \\ &= e_{n-i+1} + \text{ higher } e\text{'s.}\end{aligned}$$

$\rightsquigarrow (\mathfrak{G}_{\omega_1}, \mathfrak{G}_{\omega_2}, \ldots, \mathfrak{G}_{\omega_k}) = (e_{n-k+1} + \text{ higher }, e_{n-k+2} + \text{ higher }, \ldots, e_n).$

K_0 OF $X_{n,k}$: PROOF



$$\rightsquigarrow (\mathfrak{G}_{\omega_1}, \mathfrak{G}_{\omega_2}, \ldots, \mathfrak{G}_{\omega_k}) = (e_{n-k+1} + \text{ higher }, e_{n-k+2} + \text{ higher }, \ldots, e_n)$$
$$= (e_{n-k+1}, e_{n-k+2}, \ldots, e_n).$$

Main Theorem

Theorem [Z.]

The K_0 ring of the variety $X_{n,k}$ is isomorphic to the generalized coinvariant algebra $R_{n,k}$. $\square$

Note

We can write down an explicit isomorphism

$$K_0(X_{n,k}) \cong \frac{\mathbb{Z}[y_1, \ldots, y_n]}{(y_1^k, \ldots, y_n^k, e_{n-k+1}, \ldots, e_n)} \xrightarrow{\cong} \frac{\mathbb{Z}[x_1, \ldots, x_n]}{(x_1^k, \ldots, x_n^k, e_{n-k+1}, \ldots, e_n)} \cong H^\bullet(X_{n,k})$$

given by $y_i = 1 - 1/x_i$.

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