# THE GROTHENDIECK GROUP OF SPANNING LINE CONFIGURATIONS

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# The Flag Variety

- $\mathbb{F}$ = algebraically closed field of characteristic zero.
- **Complete flag**

$$E_\bullet : 0 = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_n = \mathbb{F}^n.$$

- **Flag variety**

$$Fl_n := Gl_n/B \cong \text{ parameter space of complete flags.}$$

*Go team Schubert!*

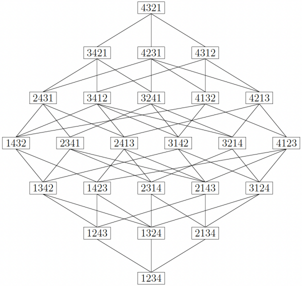
# THE FLAG VARIETY: COHOMOLOGY

**Theorem [Bor53]**

**Borel presentation.**

$$H^\bullet(Fl_n;\mathbb{Z}) \cong CH^\bullet(Fl_n) \cong \frac{\mathbb{Z}[x_1,x_2,\ldots,x_n]}{(e_1,e_2,\ldots,e_n)}.$$

- Affine stratification by **Schubert varieties** $X_w$ for $w \in \mathcal{S}_n$.
- **Schubert polynomial** $\mathfrak{S}_w = [X_w]$.
- Poset of cells = **Bruhat order**.




Bruhat order for $\mathcal{S}_4$ [BGPng].

# The Flag Variety: Grothendieck Group

**Theorem [AH61]**

$$K_0(Fl_n) \cong \frac{\mathbb{Z}[x_1, x_2, \ldots, x_n]}{(e_1, e_2, \ldots, e_n)}.$$

- $(K_0(X), \oplus, \otimes)$ – ring of (honest & virtual) vector bundles over $X$.

- $K_0(Fl_n) \cong CH^\bullet(Fl_n) \cong H^\bullet(Fl_n)$.

- **Grothendieck polynomial** $\mathfrak{G}_w = [\mathcal{O}_{X_w}]$.

# THE FLAG VARIETY: COMBINATORICS

Motivating example:

Schubert classes in $Gr(k,n) \rightsquigarrow$ Schur symmetric polynomials $s_\lambda$.

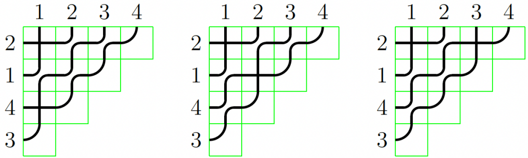
$$s_\lambda = \sum_{T \in \mathrm{SSYT}(\lambda)} x^T.$$

$$s_{\ydiagram{2,1,1}} = x_1^2 x_2 x_3 + x_1 x_2^2 x_3 + x_1 x_2 x_3^2$$

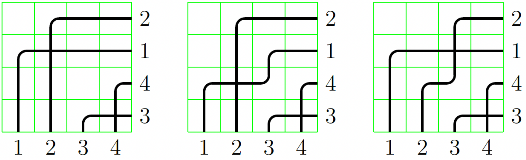
$$\begin{ytableau} 1 & 1 \\ 2 \\ 3 \end{ytableau} \qquad \begin{ytableau} 1 & 2 \\ 2 \\ 3 \end{ytableau} \qquad \begin{ytableau} 1 & 3 \\ 2 \\ 3 \end{ytableau}$$

# The Flag Variety: Combinatorics

## What are Schubert & Grothendieck polynomials counting?




*Reduced pipe dreams* for $w = 2143$ [BGPng]




*Bumpless pipe dreams* for $w = 2143$ [BGPng]

# Spanning Line Configurations

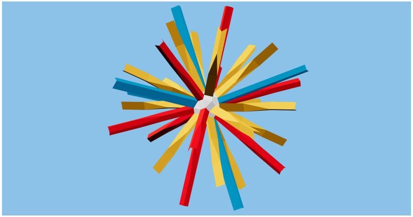
- $\mathbb{F}$ = algebraically closed field of characteristic zero.

- **Spanning line configuration**

$$\ell_\bullet = (\ell_1, \ldots, \ell_n) \mid \ell_i \text{ line in } \mathbb{F}^k, \operatorname{span} \ell_\bullet = \mathbb{F}^k.$$

- The variety of spanning line configurations

$$\begin{aligned} X_{n,k} &:= \mathrm{Mat}^\circ_{\text{full rank}} / T \\ &\cong \text{ parameter space of spanning line configurations.} \end{aligned}$$



# Spanning Lines: $H^\bullet, CH^\bullet, K_0$

**Theorem [PR19]**

**Borel-style presentation.**

$$H^\bullet(X_{n,k}) \cong \frac{\mathbb{Z}[x_1, \ldots, x_n]}{(x_1^k, \ldots, x_n^k, e_{n-k+1}, \ldots, e_n)}.$$

**Theorem (Z. '25)**

$$K_0(X_{n,k}) \cong CH^\bullet(X_{n,k}) \cong \frac{\mathbb{Z}[x_1, \ldots, x_n]}{(x_1^k, \ldots, x_n^k, e_{n-k+1}, \ldots, e_n)}.$$

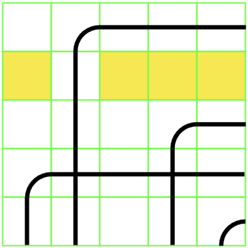
# Spanning Lines: Combinatorics

- Affine stratification by **Fubini words** $w : [n] \twoheadrightarrow [k]$.

  $235125 \in \mathrm{Words}(5,3)$ non-Fubini | $235145$ is Fubini

- **Schubert & Grothendieck polynomials** for words.

- Bumpless pipedreams?



# References I


[AH61] M. F. Atiyah and Friedrich Hirzebruch. “Vector Bundles and Homogeneous Spaces”. In: *Proceedings of the Symposium on Pure Mathematics*. Vol. 3. Symposia in Pure Mathematics. American Mathematical Society, 1961, pp. 7–38.

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[Bor53] Armand Borel. “Sur La Cohomologie des Espaces Fibres Principaux et des Espaces Homogenes de Groupes de Lie Compacts”. In: *Annals of Mathematics* 57 (1953), p. 115. URL: `https://api.semanticscholar.org/CorpusID:123804171`.

[PR19] Brendan Pawlowski and Brendon Rhoades. “A Flag Variety for the Delta Conjecture”. In: *Transactions of the American Mathematical Society* 372.11 (2019). Article electronically published on September 12, 2019, pp. 8195–8248. DOI: `10.1090/tran/7918`. URL: `https://doi.org/10.1090/tran/7918`.