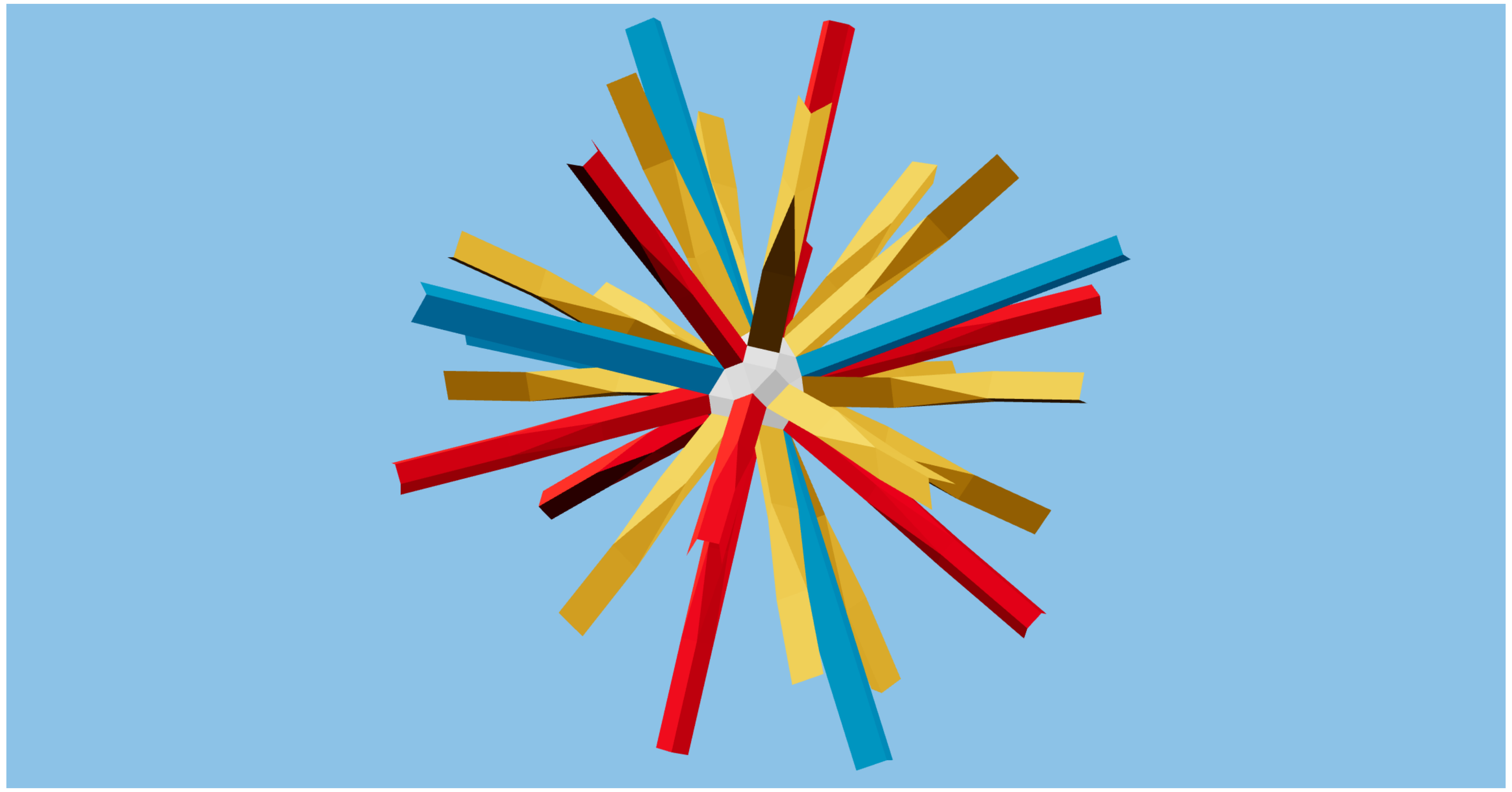
K_0-Ring of Spanning Line Configurations

& Combinatorics of Fubini Words



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Background: The Flag Variety

Consider the action $\mathcal{S}_n \curvearrowright \mathbb{Z}[x_1, \ldots, x_n]$.

The *coinvariant algebra* is

$$R_n := \frac{\mathbb{Z}[x_1, \ldots, x_n]}{\textit{Hilbert ideal of } \mathcal{S}_n\textit{-invariants}} = \frac{\mathbb{Z}[x_1, \ldots, x_n]}{(e_1, \ldots, e_n)}.$$

The *flag variety* parametrizes *complete flags* $E_\bullet : 0 = E_0 \subseteq \cdots \subseteq E_n = \mathbb{F}^n$.

$$Fl_n \cong GL_n/B$$

Cell decomposition with *Schubert cells* $C_\omega, \quad \omega \in \mathcal{S}_n$. *Schubert varieties* $X_\omega := \overline{C}_\omega$.

Cohomologies of the Flag Variety

The *Grothendieck group* $K_0(X) \ := \ \dfrac{\mathbb{Z} \cdot \operatorname{Iso} \mathrm{VB}_X}{[\mathscr{E}] = [\mathscr{E}'] + [\mathscr{E}''] \text{ if } \mathscr{E}' \to \mathscr{E} \to \mathscr{E}'' \text{ is SES}}.$

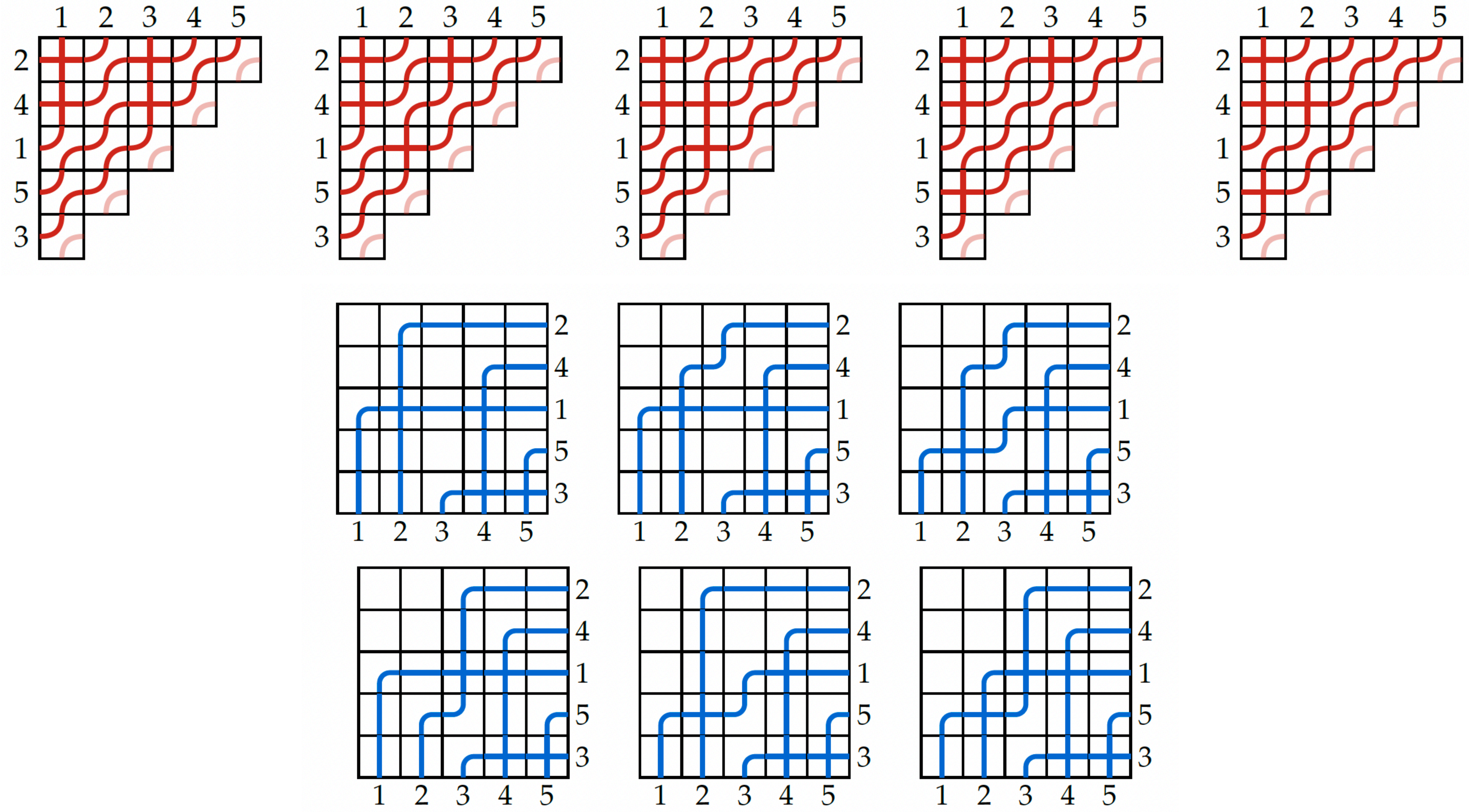
The *Chow ring* $CH^\bullet(X) \ := \ \dfrac{\textit{algebraic cycles}}{\textit{rational equivalence}}.$

Theorem (Borel Presentation [Bor53], [Dem73])

There are isomorphisms

$$K_0(Fl_n) \ \cong \ CH^\bullet(Fl_n) \ \cong \ H^\bullet(Fl_n(\mathbb{C}); \mathbb{Z}) \ \cong \ R_n .$$

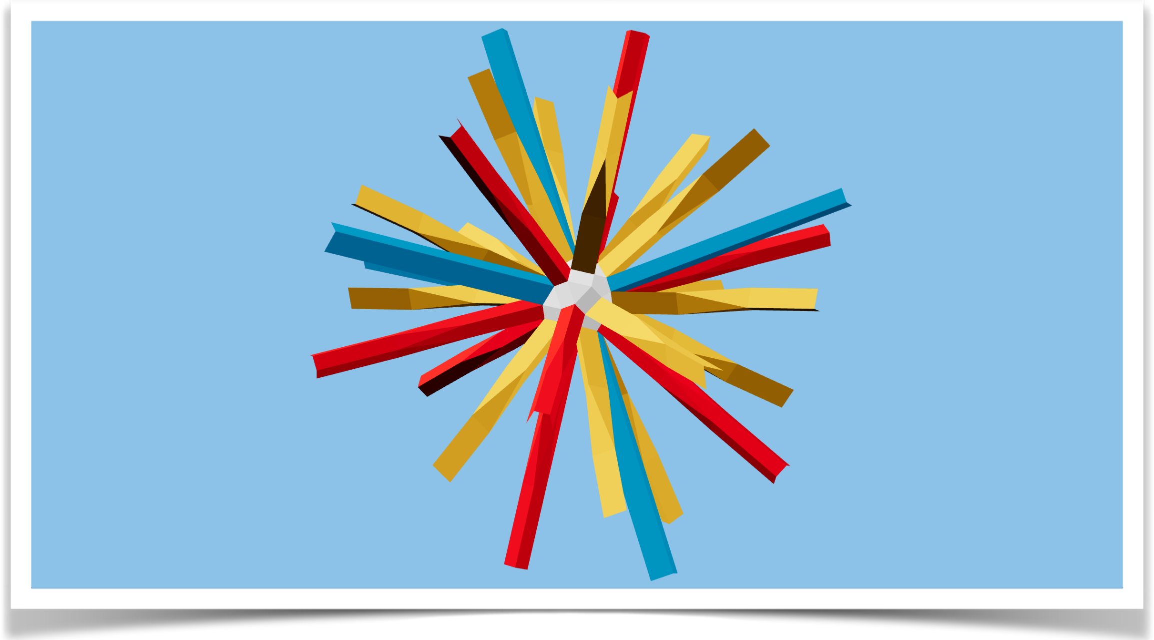
Classical and Bumpless Pipe Dreams



A Table of Generalizations

Coinvariant algebra	Generalized coinvariant algebra	$R_{n,k} := \dfrac{\mathbb{Z}[x_1, \ldots, x_n]}{(x_1^k, \ldots, x_n^k, e_{n-k+1}, \ldots, e_n)}$
Flag variety	The variety of spanning line configurations	$X_{n,k} := \{(\ell_1, \ldots, \ell_n) \mid \ell_1 + \cdots + \ell_n = \mathbb{F}^k\}$
Permutations	Fubini words, aka ordered set partitions	Surjective functions $[n] \twoheadrightarrow [k]$
Pipe dreams	Pipe dreams of a word	Truncated & decorated PD

The Variety of Spanning Line Configurations



The *generalized coinvariant algebra* [HRS18] *is*

$$R_{n,k} \;:=\; \frac{\mathbb{Z}[x_1, \ldots, x_n]}{(x_1^k, \ldots, x_n^k, e_{n-k+1}, \ldots, e_n)}.$$

The *variety of spanning line configurations* [PR19] $X_{n,k}$ parametrizes

$$\ell_\bullet = ([\ell_1], \ldots, [\ell_n]) \;\in\; (\mathbb{P}^{k-1})^n, \quad \ell_1 + \cdots + \ell_n = \mathbb{F}^k\,.$$

Cell decomposition with *Fubini words* — surjective functions $[n] \twoheadrightarrow [k]$

$$w = 21231 \quad\leftrightarrow\quad \{\ 2{,}5 \mid 1{,}3 \mid 4\ \}$$

Cohomologies of Spanning Line Configurations

Theorem [PR19]

There is an isomorphism

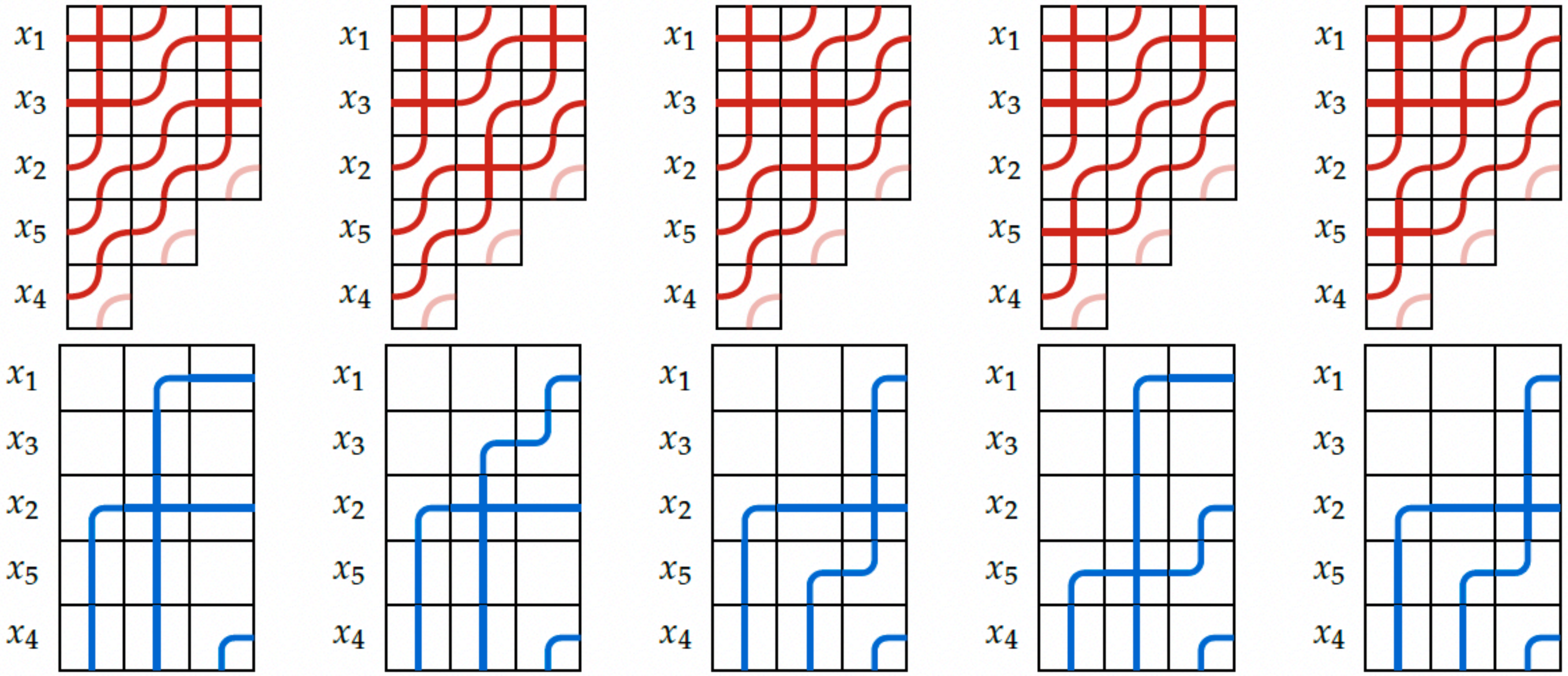
$$H^\bullet(X_{n,k}(\mathbb{C}); \mathbb{Z}) \;\cong\; R_{n,k}\,.$$

Theorem (Z. 2025+)

There are isomorphisms

$$K_0(X_{n,k}) \;\cong\; CH^\bullet(X_{n,k}) \;\cong\; H^\bullet(X_{n,k}(\mathbb{C}); \mathbb{Z}) \;\cong\; R_{n,k}\,.$$

Pipe Dreams of Fubini Words



Open Problems

Question

What are some necessary conditions on X that would imply $K_0(X) \cong CH^\bullet(X)$?

Question

Larson-Li-Payne-Proudfoot [Lar+24] proved that $K_0(X) \cong CH^\bullet(X)$ canonically when X is a wonderful variety, in the sense of [CP95].

Is $X_{n,k}$ related to a wonderful compactification?

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Based on a paper in preparation titled *The Grothendieck Group of the Variety of Spanning Line Configurations* (Zeng, 2025+).

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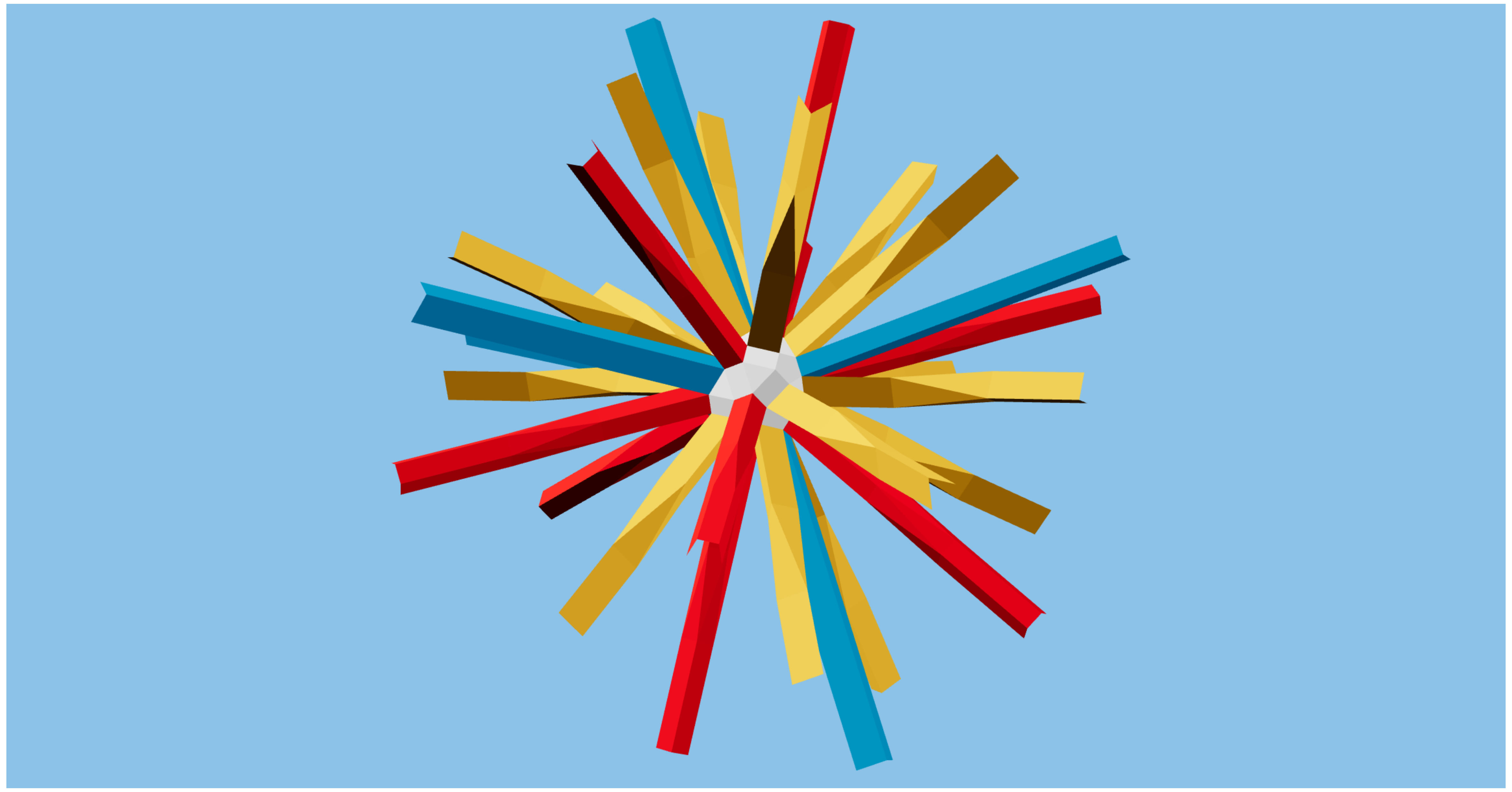

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