Gromov-Witten Invariants and Quantum Cohomology

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1 How many degree d rational curves pass through $3d-1$ general points in the projective plane?

A rational plane curve of degree d is parametrized by an embedding

$$\mathbb{P}^1 \to \mathbb{P}^2, \quad (s:t) \mapsto (f(s,t) : g(s,t) : h(s,t))$$

where f, g, h are homogeneous of degree d, each having the form

$$a_0 s^d t^0 + a_1 s^{d-1} t^1 + \cdots + s^0 t^d.$$

Let's count dimensions. Modding out scalars:

$$3(d+1) - 1 = 3d + 2.$$

Modding out reparametrizations by PGL_3:

$$3(d+1) - 1 - 3 = 3d - 1.$$

Therefore, we expect the locus of degree d rational plane curves passing through $3d-1$ general points to be *zero-dimensional*.

> **Problem 1.1.** How many degree d rational curves pass through $3d-1$ general points in the projective plane?

Call this number N_d.

Example 1.2. Two points determine a unique line.

$$N_1 = 1$$

Example 1.3. 5 points determine a unique conic.

$$N_2 = 1$$

Example 1.4. Steiner 1848:

$$N_3 = 12$$

Example 1.5. Zeuthen 1873:

$$N_4 = 620$$

Example 1.6. '80s:

$$N_5 = 87304$$

Example 1.7. Kontsevich's recursive formula 1994:

$$N_d = \sum_{d_A+d_B=d} N_{d_A} N_{d_B} d_A^2 d_B \left(d_B \binom{3d-4}{3d_A-2} - d_A \binom{3d-4}{3d_A-1} \right)$$

$\rightsquigarrow$ Recursive structure on the boundary of $\overline{M}_{0,3d-1}\left(\mathbb{P}^2, d\right)$.

Useful references:

- (Kock and Vainsencher 2003) Kontsevish's Formula for Rational Plane Curves. http://www.dmat.ufpe.br/~israel/kontsevich.html
- *"FP-Notes"* (Fulton and Pandharipande 1997) Notes on Stable Maps and Quantum Cohomology. https://arxiv.org/abs/alg-geom/9608011
- (Abramovich 2006) Lectures on Gromov-Witten Invariants of Orbifolds. https://arxiv.org/abs/math/0512372
- (Clader 2024) Curve Counting and Mirror Symmetry. https://www.ams.org/journals/notices/202409/noti3022/noti3022.html?adat=October%202024&trk=3022&pdfissue=202409&pdffile=rnoti-p1140.pdf&cat=none&type=.html
- (Pandharipande and Thomas 2016) 13/2 Ways of Counting Curves. https://arxiv.org/abs/1111.1552

2 Moduli of Stable Curves

2.1 Motivation

How is Kontsevich's method different from the ones before?

	Previous	Kontsevich
What's being counted	*honest subvarieties*	*parametrizations up to isomorphism*
Moduli space	*Severi varieties*	*Moduli of* ***stable maps***
Incidence condition	*Loci in the moduli space*	***Marked points*** *on stable maps*
Key technique	*Localization*	*Recursive structure on the boundary*

$\rightsquigarrow$ Different compactifications of the smooth locus.

$\rightsquigarrow$ Kontsevich's method does not require a full understanding of the Chow ring of the moduli space.

Slogan. Slogan: Stable stuff has *finite automorphisms*.

- Stable rational curves have NO (non-trivial) automorphisms, so we get rid of ambiguity of parametrizations.
- Get a *fine moduli space* with a *universal family*.
- In general, still get a *coarse moduli space*.

2.2 Stable Curves

Definition 2.1. An n**-pointed smooth rational curve**

$$(C, p_1, \dots, p_n)$$

is a projective smooth rational curve C equipped with a choice of n distinct points $p_1, \dots, p_n \in C$, called the marks.

An isomorphism between two n-pointed rational curves

$$\varphi : (C, p_1, \dots, p_n) \xrightarrow{\sim} (C', p'_1, \dots, p'_n)$$

is an isomorphism $\varphi : C \xrightarrow{\sim} C'$ which respects the marks (in the given order), i.e.,

$$\varphi(p_i) = p'_i, \quad i = 1, \dots, n$$

Proposition 2.2. *For $n \geqslant 3$, there is a fine moduli space $M_{0,n}$ for **n-pointed smooth rational curves** up to isomorphism.*

Example 2.3. $n = 3$. Classical fact: $2 + 1 = 3$ points uniquely determines a projective transformation over $\mathbb{P}^1$. Thus,

$$(C, p_1, p_2, p_3) \mapsto (\mathbb{P}^1, 0, 1, \infty)$$

$$\rightsquigarrow M_{0,3} \cong \text{pt}.$$

Example 2.4. $n = 4$.

$$(C, p_1, p_2, p_3, p_4) \mapsto (\mathbb{P}^1, 0, 1, \infty, \tilde{p}_4)$$

$$\rightsquigarrow M_{0,4} \cong \mathbb{P}^1 \backslash \{0, 1, \infty\}.$$

Problem 2.5. Not compact!

$\rightsquigarrow$ Needs to introduce degenerations.

Definition 2.6. A **tree** of projective lines is a connected curve such that

- Each irreducible component (*twig*) is isomorphic to a projective line.
- The points of intersection of the components are ordinary double points.
- There are no closed circuits. That is, if a node is removed, the curve becomes disconnected.

The three properties together are equivalent to saying that the curve has arithmetic genus zero.

Definition 2.7. $n \geqslant 3$. A **stable n-pointed rational curve** is a tree C of projective lines, with n distinct marks which are smooth points of C, such that every twig has at least three special points (either marks or nodes).

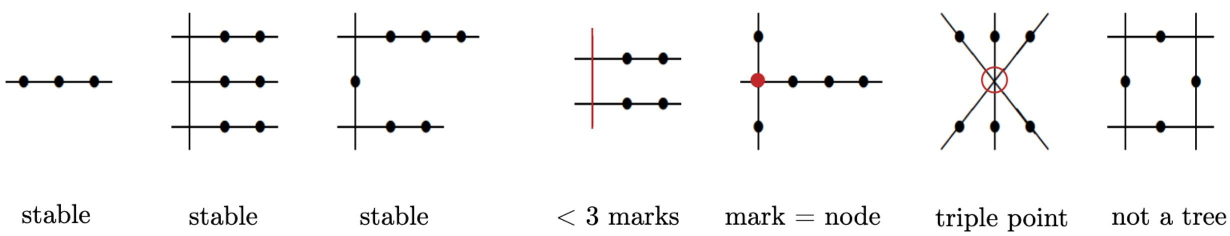


Figure 2.8: Examples and non-examples of stable curves.

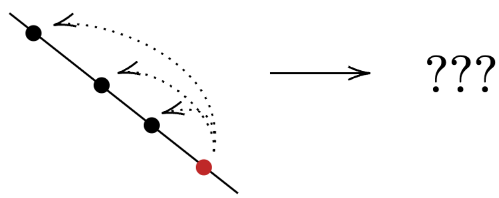
Remark 2.9. Stable $\rightsquigarrow$ no automorphisms.

Theorem 2.10. *(Knudsen) For each $n \geqslant 3$, there is a fine moduli space $\overline{M}_{0,n}$ for stable n-pointed rational curves. It is a projective variety and contains the subvariety $M_{0,n}$ as a dense open subset.*

Example 2.11. $\overline{M}_{0,4}$ is a compactification of $M_{0,4} \cong \mathbb{P}^1 \backslash \{0, 1, \infty\}$. In fact,

$$\overline{M}_{0,4} \cong \mathbb{P}^1.$$

What are the three new degeneracies corresponding to $0, 1, \infty$? In other words, what happens when p_4 approaches $0, 1, \infty$?



> **Slogan.** When you can't tell 2 points apart, blow the thing up!

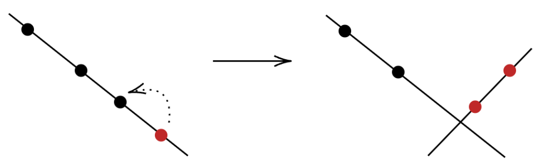


Figure 2.12: The '0', '1', and '∞' are stable curves with an additional twig.

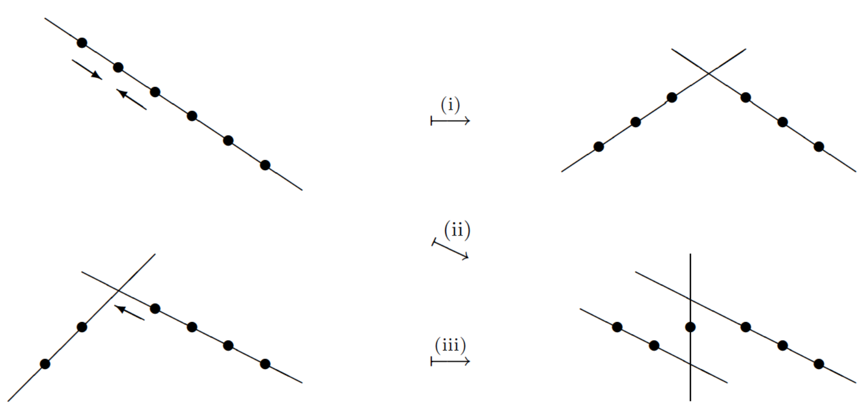


Figure 2.13: Other instances from Kock.

2.3 Stabilization

⇝ Adding and forgetting marked points may break stability. ⇝ Stabilize the new curve by blowing up / down.

Example 2.14. Adding a new mark q.

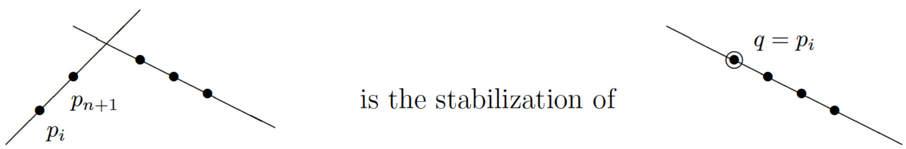


Figure 2.15: Blow up if q coincides with a previous mark.

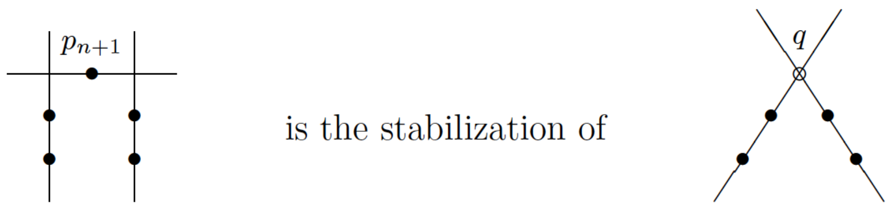


Figure 2.16: Blow up if q coincides with a node.

Example 2.17. Forgetting p_{n+1}.

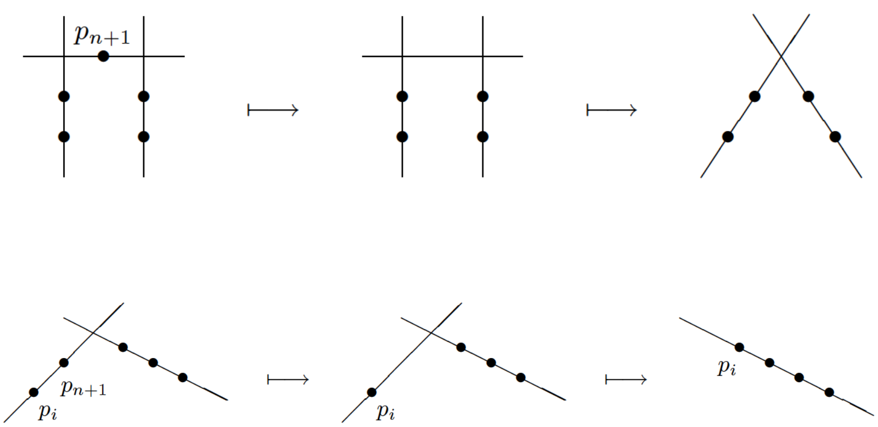


Figure 2.18: Contract the twig with < 3 marked points.

Proposition 2.19. *The map*

$$\pi : \overline{M}_{0,n+1} \to \overline{M}_{0,n}$$

forgetting p_{n+1} is a morphism.

It is actually something more.

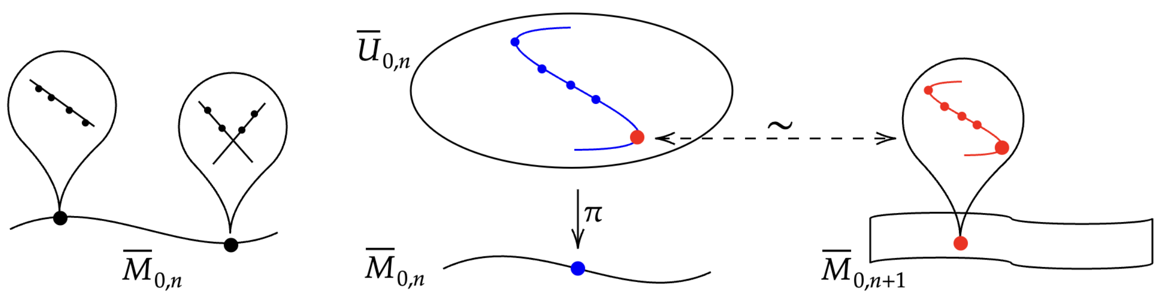
Theorem 2.20. *The forgetful morphism*

$$\pi : \overline{M}_{0,n+1} \to \overline{M}_{0,n}$$

is the universal family $\overline{U}_{0,n}$ over $\overline{M}_{0,n}$.

Proof

Idea: points in $\overline{U}_{0,n}$ bijectively corresponds to stable $(n+1)$-pointed curves.



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2.4 Boundary Divisors

Twigs partition marks into disjoint subsets.

The boundary of $\overline{M}_{0,n}$ is stratified by set partitions of marked points, ordered by refinement.

Example 2.21. Stratification for $\overline{M}_{0,6}$. Numbers on the side count different orderings of the 6 marked points.

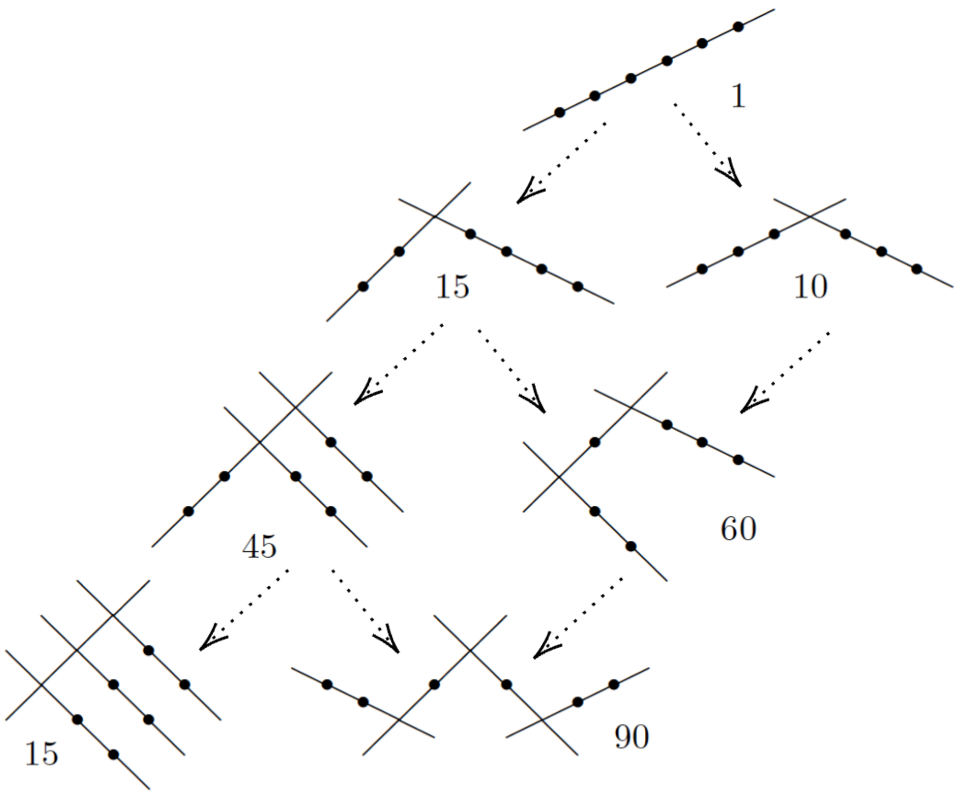


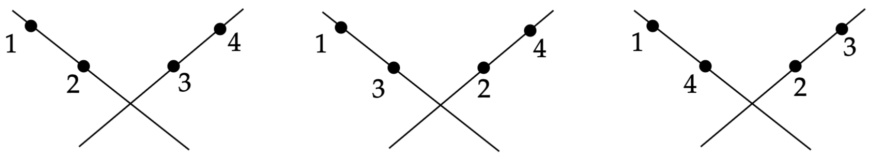
Figure 2.22: The poset structure of $\overline{M}_{0,6}$, cf. Kock

Definition 2.23. Denote by $[n]$ the set of n marks. There is an irreducible **boundary divisor** $D(A \mid B)$ for each partition $[n] = A \sqcup B$ with A, B each having at least 2 elements.
$\rightsquigarrow$ Points in $D(A \mid B)$ correspond to curves with two twigs and specified marked points.
$\rightsquigarrow$ Intersecting boundary divisors corresponds to taking common refinement of set partitions.

Example 2.24. $n = 4$. There are three set partitions of $[4]$:

$$(12 \mid 34), (13 \mid 24), (14 \mid 23)$$

This gives rise to the three boundary divisors of $\overline{M}_{0,4}$:



$D(12 \mid 34)$ $D(13 \mid 24)$ $D(14 \mid 23)$

Recall that $\overline{M}_{0,4} \cong \mathbb{P}^1$, where every two points are linearly equivalent. Therefore,

$$D(12 \mid 34) \sim D(13 \mid 24) \sim D(14 \mid 23) \text{ in } \overline{M}_{0,4}. \tag{$*$}$$

Now, consider the morphism

$$\pi : \overline{M}_{0,n} \to \overline{M}_{0,ijkl}, \quad (C, p_1, \ldots, p_n) \mapsto (C, p_i, p_j, p_k, p_l).$$

This is a composition of forgetful maps which remembers any given quadruplet of points labeled i, j, k, l. Pulling back relation $(*)$ along π, we obtain the **fundamental relations** for $\overline{M}_{0,n}$:

Proposition 2.25 (Fundamental relations).

$$\sum_{\substack{i,j \in A \\ k,l \in B}} D(A \mid B) = \sum_{\substack{i,k \in A \\ j,l \in B}} D(A \mid B) = \sum_{\substack{i,l \in A \\ j,k \in B}} D(A \mid B) \quad \textit{in } A^1(\overline{M}_{0,n}).$$

Abbreviating,

$$D(ij \mid kl) = D(ik \mid jl) = D(il \mid kj).$$

*(call these sums **special boundary divisors**.)*

In fact, these completely describe the Chow ring:

Theorem 2.26 (Keel). *The Chow ring $A(\overline{M}_{0,n})$ is generated by classes of boundary divisors $D(A \mid B)$, with the following relations.*

- *For two partitions $(A_1 \mid B_1)$ and $(A_2 \mid B_2)$ with no inclusions among $A_1, \ldots, B_2$,*

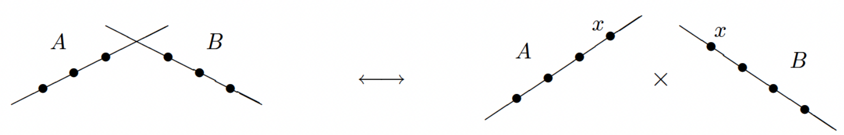
$$[D(A_1 \mid B_1)] \cdot [D(A_2 \mid B_2)] = 0.$$

- *The fundamental relations.*

Proposition 2.27 (Recursive structure). *There is a canonical isomorphism*

$$D(A \mid B) \simeq \overline{M}_{0,A\cup\{x\}} \times \overline{M}_{0,B\cup\{x\}}$$

for which x is the unique node on a two-twigged curve.



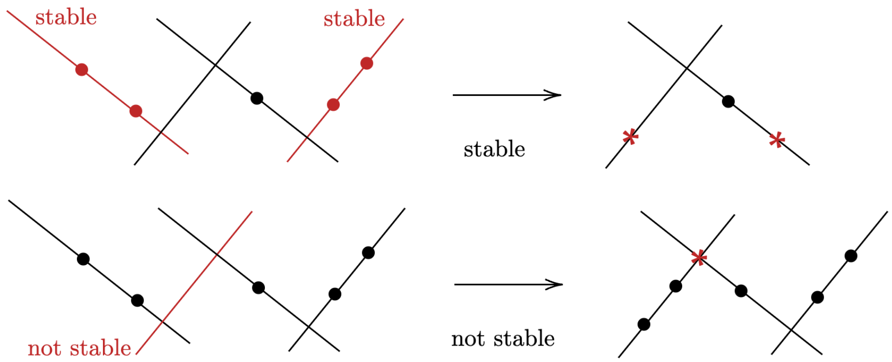
3 Moduli of Stable Maps

3.1 Stable Maps

Definition 3.1. An n**-pointed map** is a morphism $\mu : C \to \mathbb{P}^r$, where C is a tree of projective lines with n distinct marks which are smooth points of C.

An n**-pointed stable map** $\mu : C \to \mathbb{P}^r$ is such a map for which any twig mapped to a point is stable as a pointed curve.

Example 3.2. The second map contracts a $\mathbb{P}^1$ with only two marks, so not stable.



$$\textit{stable map} \iff \textit{finite automorphisms}$$

Theorem 3.3. *There exists a coarse moduli space $\overline{M}_{0,n}(\mathbb{P}^r, d)$ parametrizing **isomorphism classes of stable n-pointed maps of degree d to** $\mathbb{P}^r$.*
$\overline{M}_{0,n}(\mathbb{P}^r, d)$ is a projective normal irreducible variety, locally isomorphic to a quotient of a smooth variety (moduli of parametrizations) by a finite group action. There exists a fine moduli space $M_{0,n}(\mathbb{P}^r, d)$ for stable maps with trivial automorphisms, and it a dense open set of $\overline{M}_{0,n}(\mathbb{P}^r, d)$.

Proposition 3.4. *The dimension of $\overline{M}_{0,n}(\mathbb{P}^r, d)$ is $(r+1)(d+1)-1-3+n$.*

3.2 Forgetful Maps and Evaluation Maps

Two kinds of forgetful maps:

- Forget points:

$$\overline{M}_{0,n+1}(\mathbb{P}^r, d) \to \overline{M}_{0,n}(\mathbb{P}^r, d).$$

- ($n \geqslant 4$) Forget that you were a map:

$$\overline{M}_{0,n}(\mathbb{P}^r, d) \to \overline{M}_{0,n}.$$

  $\rightsquigarrow$ Just take the source tree of lines with n marks and stabilize.

A new species: **evaluation maps**

$$\mathrm{ev}_i : \overline{M}_{0,n}(\mathbb{P}^r, d) \to \mathbb{P}^r, \qquad (\mu : (C, p_1, \ldots, p_n) \to \mathbb{P}^r) \mapsto \mu(p_i).$$

Proposition 3.5. *All evaluation maps and most forgetful maps ($n \geqslant 4$) are flat.*

"**Total evaluation map**"

$$\underline{\mathrm{ev}} : \overline{M}_{0,n}(\mathbb{P}^r, d) \to \mathbb{P}^r \times \cdots \times \mathbb{P}^r, \qquad \mu \mapsto (\mu(p_1), \ldots, \mu(p_n)).$$

Example 3.6. (Flat pullbacks.) Let D be a divisor on $\mathbb{P}^r$. Then, the pullback of D along ev_i is a divisor on $\overline{M}_{0,n}(\mathbb{P}^r, d)$.

Let V be a subvariety of $\mathbb{P}^r$. If $\pi : \overline{M}_{0,n+1}(\mathbb{P}^r, d) \to \overline{M}_{0,n}(\mathbb{P}^r, d)$ is the map forgetting p_{n+1}, then

$$\mathrm{inc}(V) := \pi \circ \mathrm{ev}^{-1}(V)$$

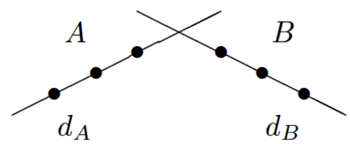
is the locus in $\overline{M}_{0,n}(\mathbb{P}^r, d)$ consisting of stable maps whose image is *incident to V*.

3.3 Boundary Divisors

$\rightsquigarrow$ Set partitions get upgraded to *weighted partitions*.

Definition 3.7. A d-weighted partition of $[n]$ consists of the data $A \cup B = [n], d_A + d_B = d$, (where $\#A \geqslant 2$ if $d_A = 0$, and $\#B \geqslant 2$ if $d_B = 0$).

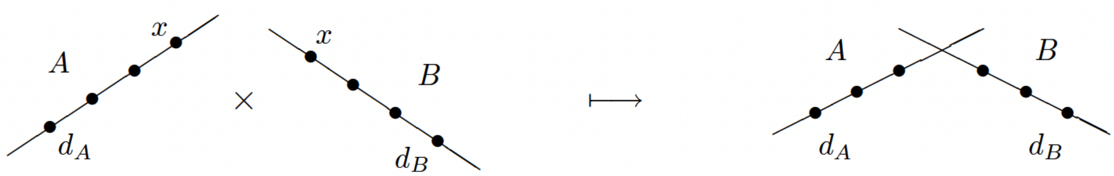
For each d-weighted partition, there exists an irreducible **boundary divisor** $D(A, B; d_A, d_B)$.



Proposition 3.8 (Recursive structure). *There is a gluing isomorphism*

$$D(A, B; d_A, d_B) \cong \overline{M}_{0,A\cup\{x\}}(\mathbb{P}^r, d_A) \times_{\mathbb{P}^r} \overline{M}_{0,B\cup\{x\}}(\mathbb{P}^r, d_B)$$

where the fiber product is taken over the evaluation maps at the node x.



Proposition 3.9 (Fundamental relations). *Pulling back along forgetful maps $\overline{M}_{0,n}(\mathbb{P}^r, d) \to \overline{M}_{0,n} \to \overline{M}_{0,4}$, we once again obtain*

$$\sum_{\substack{A\cup B=[n] \\ i,j\in A \\ k,l\in B \\ d_A+d_B=d}} D(A, B; d_A, d_B) = \sum_{\substack{A\cup B=[n] \\ i,k\in A \\ j,l\in B \\ d_A+d_B=d}} D(A, B; d_A, d_B) = \sum_{\substack{A\cup B=[n] \\ i,l\in A \\ j,k\in B \\ d_A+d_B=d}} D(A, B; d_A, d_B).$$

In terms of special boundary divisors, we once again have

$$D(ij \mid kl) = D(ik \mid jl) = D(il \mid kj).$$

Remark 3.10. A few cute examples.

- $\overline{M}_{0,0}(\mathbb{P}^r, 1) \cong \mathbb{G}(1, r)$.
- $\overline{M}_{0,1}(\mathbb{P}^r, 1) \cong \mathbb{P}(U)$ for the universal bundle of $\mathbb{G}(1, r)$.
- $\overline{M}_{0,2}(\mathbb{P}^r, 1)$ is isomorphic to $\mathbb{P}^r \times \mathbb{P}^r$ blown up along the diagonal.
- $\overline{M}_{0,n}(\mathbb{P}^r, 0) \cong \overline{M}_{0,n} \times \mathbb{P}^r$.
- $\overline{M}_{0,0}(\mathbb{P}^r, 2)$ is isomorphic to the space of *complete conics*.

4 Gromov-Witten Invariants and Kontsevich's Recursive Formula

4.1 Kontsevich's Formula

Theorem 4.1 (Kontsenich's Formula). *Let N_d be the number of rational curves of degree d passing through $3d-1$ general points in the plane. Then the following recursive relation holds:*

$$N_d + \sum_{\substack{d_A+d_B=d \\ d_A\geqslant 1, d_B\geqslant 1}} \binom{3d-4}{3d_A-1} d_A^2 N_{d_A}\cdot N_{d_B}\cdot d_A d_B = \sum_{\substack{d_A+d_B=d \\ d_A\geqslant 1, d_B\geqslant 1}} \binom{3d-4}{3d_A-2} d_A N_{d_A}\cdot d_B N_{d_B}\cdot d_A d_B$$

Since we know $N_1 = 1$, the formula allows for the computation of any N_d.

Proof

A consequence of the **fundamental relation**

$$D(qr \mid st) = D(qs \mid rt).$$

To get numbers out of a linear equivalence of divisors, we hope to intersect both sides with some curve.

Take $n = 3d$ marked points, $d \geqslant 2$. Label the marked points in a convenient way:

$$\{1, 2, \ldots, n-4, q, r, s, t\}.$$

Take general lines ℓ_q, ℓ_r and $n-2$ other general points $p_1, \ldots, p_{n-4}, p_s, p_t$. In $\overline{M}_{0,n}\left(\mathbb{P}^2, d\right)$, consider the incidence locus

$$Y := \mathrm{ev}_1^{-1}(p_1) \cap \cdots \cap \mathrm{ev}_{n-4}^{-1}(p_{n-4}) \cap \mathrm{ev}_q^{-1}(\ell_q) \cap \mathrm{ev}_r^{-1}(\ell_r) \cap \mathrm{ev}_s^{-1}(p_s) \cap \mathrm{ev}_t^{-1}(p_t).$$

Namely, Y is the locus where the image of the stable pointed map is incident to all given $n-2$ points and 2 lines. By Bertini, Y is a non-singular curve contained in the

automorphism-free locus $M_{0,n}\left(\mathbb{P}^2, d\right)$, intersecting boundary divisors transversally at general points on the boundary (i.e. no excess intersections).
$\rightsquigarrow Y \cap D(A,B;d_A,d_B)$ should be a finite set of points!

We now claim that the formula follows from

$$\#Y \cap D(qr \mid st) = \#Y \cap D(qs \mid rt).$$

First,

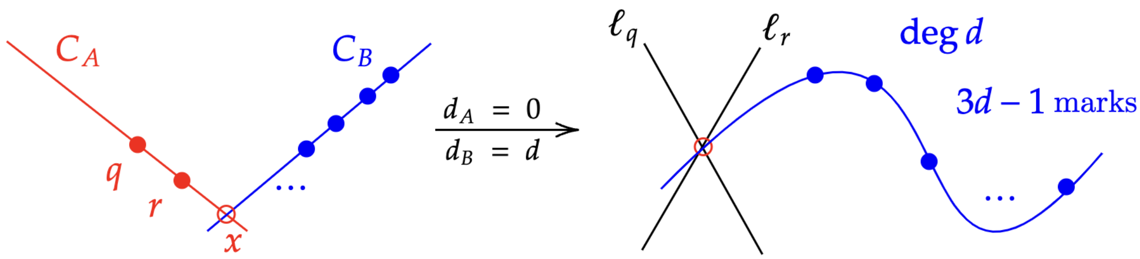
$$D(qr \mid st) = \sum_{q,r\in A, s,t\in B, d_A+d_B=d} D\left(A,B;d_A,d_B\right).$$

Examine terms in this sum:

<u>Special Case I $d_A = 0; d_B = d$.</u>

- $Y \cap D(A,B;d_A,d_B) \neq \varnothing$ iff $A = \{q,r\}$.
- $d_A = 0$ means the A-twig C_A gets contracted to the node x, which then has to land on the point of intersection $\ell_q \cap \ell_r$.
- $B =$ everything but q, r means the B-twig C_B gets $3d-2$ marked points.
- In total, the image is a degree d rational curve with $3d-2+$ node $= 3d-1$ marked points!

$$\rightsquigarrow \#Y \cap D(\{q,r\}, \text{ the rest }; 0, d) = N_d$$



Special Case II $d_A = d; d_B = 0$.

$\rightsquigarrow$ Empty intersection.

Everything else $1 \leqslant d_A \leqslant n-1$.

- $Y \cap D(A, B; d_A, d_B) \neq \varnothing$ iff $|A| = 3d_A + 1$.
- Number of set partitions $(A \mid B)$ s.t. $|A| = 3d_A + 1$, $q, r \in A$, $s, t \in B$:

$$\binom{3d-4}{3d_A-1}.$$

- **Recursive structure:** Image of C_A is a $(3d_A - 1)$-pointed rational curve of degree d_A. Image of C_B is a $(3d_B - 1)$-pointed rational curve of degree d_B.

$$\rightsquigarrow \text{ Factor of } N_{d_A} N_{d_B}.$$

- Image of q, r lands on intersections of the image of C_A with ℓ_q, ℓ_r, respectively.

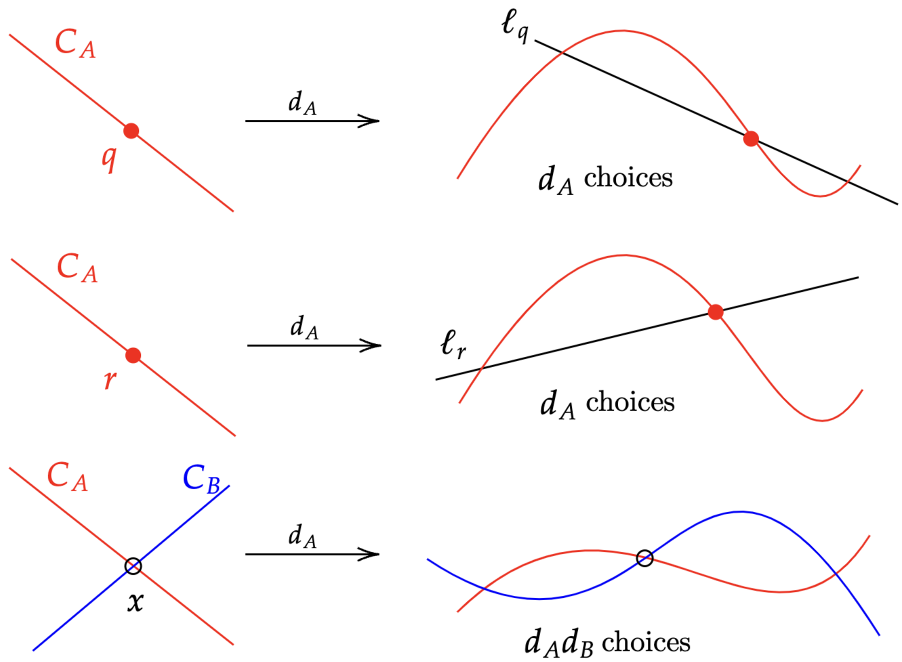
$$\rightsquigarrow \text{ Factor of } d_A^2.$$

- Image of the node $x = C_A \cap C_B$ lands on intersections of images of C_A, C_B.

$$\rightsquigarrow \text{ Factor of } d_A d_B.$$

- In total for each such divisor,

$$\#Y \cap D(A, B; d_A, d_B) = d_A^3 d_B N_{d_A} N_{d_B} \binom{3d-4}{3d_A-1}.$$



Thus,

$$\text{LHS} = N_d + \sum_{d_A+d_B=d, d_A\geqslant 1, d_B\geqslant 1} N_{d_A} N_{d_B} d_A^3 d_B \binom{3d-4}{3d_A-1}.$$

Next,

$$D(qs \mid rt) = \sum_{q,s\in A, r,t\in B, d_A+d_B=d} D\left(A,B;d_A,d_B\right).$$

Examine terms in this sum:

<u>Special Case I $d_A = 0; d_B = d.$</u>

$\rightsquigarrow$ Empty intersection.

<u>Special Case II $d_A = d; d_B = 0.$</u>

$\rightsquigarrow$ Empty intersection.

<u>Everything else $1 \leqslant d_A \leqslant n-1.$</u>

$$\#Y \cap D(A,B;d_A,d_B) = N_{d_A} N_{d_B} d_A^2 d_B^2 \binom{3d-4}{3d_A-2}.$$

So

$$\text{RHS} = \sum_{d_A+d_B=d, d_A\geqslant 1, d_B\geqslant 1} N_{d_A} N_{d_B} d_A^2 d_B^2 \binom{3d-4}{3d_A-2}.$$

All in all,

$$\begin{aligned} N_d \qquad &+ \sum_{d_A+d_B=d, d_A\geqslant 1, d_B\geqslant 1} N_{d_A} N_{d_B} d_A^3 d_B \binom{3d-4}{3d_A-1} \\ &= \sum_{d_A+d_B=d, d_A\geqslant 1, d_B\geqslant 1} N_{d_A} N_{d_B} d_A^2 d_B^2 \binom{3d-4}{3d_A-2}. \end{aligned}$$

■

4.2 Gromov-Witten Invariants

Let X be a *convex* variety (i.e. having enough lines: $\mathbb{P}^r$, Grassmannians, the Flag variety, etc., and products thereof).

- $\mathcal{T}_X$ globally generated.
- $A(X)$ has a natural basis consisting of Schubert classes.

Fix an effective class $\beta \in A_1(X)$. We can consider the moduli space of stable n-pointed maps $\mu : C \to X$ whose image $[\mu(C)] = \beta$:

$$\overline{M}_{0,n}(X, \beta).$$

We may similarly integrate pullback classes of the appropriate codimension and expect the result to be a number.

Definition 4.2. Let $\gamma_1, \ldots, \gamma_n \in A^\bullet(X)$ be classes such that

$$\sum \operatorname{codim}(\gamma_i) = \dim X + \int_\beta c_1(T_X) + n - 3.$$

Then, the **Gromov-Witten invariant** is defined as

$$I_\beta(\gamma_1 \cdots \gamma_n) = \int_{\overline{M}_{0,n}(X,\beta)} \mathrm{ev}_1^*(\gamma_1) \cup \cdots \cup \mathrm{ev}_n{}^*(\gamma_n).$$

This is an integral over the (virtual) fundamental class $\overline{M}_{0,n}(X, \beta)$. In the case that γ_i all correspond to honest subvarieties Γ_i, the Gromov-Witten invariant then counts the number of curves in the cycle class β which are incident to all Γ_i.

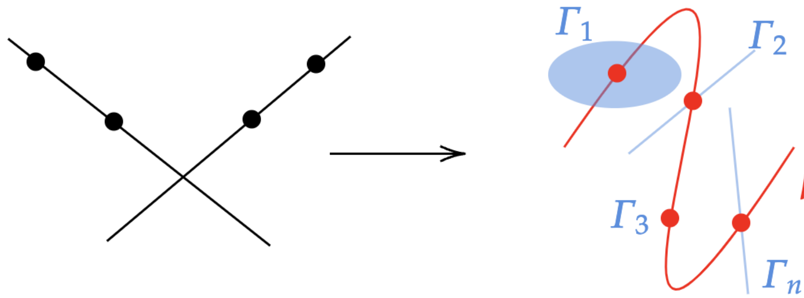


Figure 4.3: Number of curves in the class β incident to $\Gamma_1, \ldots, \Gamma_n$.

For $X = \mathbb{P}^r$ (or $\operatorname{Pic} X \cong \mathbb{Z}$), by I_d and $\overline{M}_{0,n}(\mathbb{P}^r, d)$ we mean $\beta = d \cdot h^{r-1}$ (d times class of a straight line).

Example 4.4. For $\mathbb{P}^2$, we have

$$I_d(\underbrace{h^2 \cdots h^2}_{3d-1 \text{ factors}}) = N_d$$

the number of rational curves of degree d which pass through $3d-1$ general points.

Example 4.5. In $\mathbb{P}^3$, the invariant $I_1\left(h^2 \cdot h^2 \cdot h^2 \cdot h^2\right)$ is the number of lines incident to four given lines.

Example 4.6. For $\mathbb{P}^3$, the number

$$I_3(\underbrace{h^2 \cdots h^2}_{6} \cdot \underbrace{h^3 \cdots h^3}_{3})$$

is the number of twisted cubics meeting 6 lines and 3 points. It is computed in the 21-dimensiona space $\overline{M}_{0,9}\left(\mathbb{P}^3, 3\right)$.

Example 4.7. ***27 lines example?

4.3 The Reconstruction Theorem

Proposition 4.8. *Useful properties of Gromov-Witten invariants on $\mathbb{P}^r$:*

- *(Mapping to a point) The only non-zero Gromov-Witten invariants with $d = 0$ are those with 3 marks and $\sum \operatorname{codim} \gamma_i = r$. In this case we have*

$$I_0 (\gamma_1 \cdot \gamma_2 \cdot \gamma_3) = \int_{\mathbb{P}^r} \gamma_1 \cup \gamma_2 \cup \gamma_3.$$

- *(Two-pointed invariants) The only non-zero Gromov-Witten invariants with less than three marks are*

$$I_1 (h^r \cdot h^r) = 1,$$

i.e. there is a unique line passing through two distinct points.

- *(**Divisor equation**) Suppose $d > 0$ and that one of the classes is the hyperplane class, say $\gamma_{n+1} = h$. Then*

$$I_d (\gamma_1 \cdots \gamma_n \cdot h) = I_d (\gamma_1 \cdots \gamma_n) \cdot d$$

The recursive structure of boundary divisors, namely

$$D (A, B; d_A, d_B) \simeq \overline{M}_{0,A \cup \{x\}} (\mathbb{P}^r, d_A) \times_{\mathbb{P}^r} \overline{M}_{0,B \cup \{x\}} (\mathbb{P}^r, d_B)$$

acquires the following new guise:

Lemma 4.9 (**Splitting lemma**). *For a boundary divisor D, let $\alpha : D \hookrightarrow \overline{M}$ be the natural inclusion, and let $\iota : D \cong (\mathrm{ev}_{x_A} \times \mathrm{ev}_{x_B})^{-1}(\Delta) \hookrightarrow \overline{M}_A \times \overline{M}_B$ be the inclusion as the inverse image of the diagonal. Then for any classes $\gamma_1, \ldots, \gamma_n \in A^*(\mathbb{P}^r)$ the following identity holds in $A^*(\overline{M}_A \times \overline{M}_B)$:*

$$\iota_* \alpha^* \underline{\mathrm{ev}}^*(\underline{\gamma}) = \sum_{e+f=r} \left(\prod_{a \in A} \mathrm{ev}_a^*(\gamma_a) \cdot \mathrm{ev}_{x_A}^*(h^e) \right) \times \left(\prod_{b \in B} \mathrm{ev}_b^*(\gamma_b) \cdot \mathrm{ev}_{x_B}^*\left(h^f\right) \right)$$

Corollary 4.10 (A general recursive formula)**.**

$$\int_D \mathrm{ev}_1^*(\gamma_1) \cup \cdots \cup \mathrm{ev}_n^*(\gamma_n) = \sum_{e+f=r} I_{d_A}\left(\prod_{a \in A} \gamma_a \cdot h^e \right) \cdot I_{d_B}\left(\prod_{b \in B} \gamma_b \cdot h^f \right).$$

All Gromov-Witten invariants for $\mathbb{P}^r$ can similarly be reconstructed from these recursions, knowing just a single initial condition.

Theorem 4.11 (**Reconstruction theorem for $\mathbb{P}^r$**)**.** *(Kontsevich-Manin and Ruan-Tian.) All the Gromov-Witten invariants for $\mathbb{P}^r$ can be computed recursively, and the only necessary initial value is $I_1(h^r \cdot h^r) = 1$, the number of lines through two points.*

d	N_d
1	1
2	1
3	12
4	620
5	87304
6	26312976
7	14616808192

First few invariants for $\mathbb{P}^2$.

5 Quantum Cohomology

We have some huge recursions and numbers that *blows up*. We should try organizing them into generating series.

5.1 An Interlude on Generating Functions

Definition 5.1. Let $\{N_k\}_{k=0}^{\infty}$ be a sequence of numbers. The **exponential generating function** for N_k is

$$F(x) := \sum_{k=0}^{\infty} \frac{x^k}{k!} N_k.$$

Slogan.

unordered objects $\rightsquigarrow$ ordinary generating functions

labeled objects $\rightsquigarrow$ exponential generating functions

Example 5.2. The derivative of an exponential generating function is a shift in its indices:

$$\frac{d}{dx} F = \sum_{k=1}^{\infty} k \frac{x^{k-1}}{k!} N_k = \sum_{k=0}^{\infty} \frac{x^k}{k!} N_{k+1}.$$

Example 5.3. Fibonacci numbers.

- Initial conditions:

$$N_0 = N_1 = 1.$$

- Recursion:

$$N_{k+2} = N_{k+1} + N_k.$$

- In terms of the generating function:

$$F_{xx} = F_x + F.$$

Slogan.

Recursive formulae $\iff$ PDE in generating functions

5.2 Quantum Cohomology and Associativity of the Quantum Product

$\rightsquigarrow$ Multi indicies: For $\mathbf{x} = (x_0, \ldots, x_r)$ and $\mathbf{a} = (a_0, \ldots, a_r)$, and define

$$\mathbf{x}^{\mathbf{a}} = x_0^{a_0} \cdots x_r^{a_r} \quad \text{and} \quad \mathbf{a}! = a_0! \cdots a_r!$$

$\rightsquigarrow$ By the '$\bullet$' in $(h^i)^{\bullet a_i}$, we mean take i many times the class h^i, which is NOT the same as taking the their product in the Chow ring.

$\rightsquigarrow$ Short hand for I_d: write

$$I(\gamma_1 \cdots \gamma_n) := \sum_{d=0}^{\infty} I_d(\gamma_1 \cdots \gamma_n),$$

since at most one of the terms is non-zero, namely when

$$d = \frac{\sum \operatorname{codim} \gamma_i - r - n + 3}{r+1}.$$

Definition 5.4. The **Gromov-Witten potential** for classes $(h^0)^{\bullet a_0}, \ldots, (h^r)^{\bullet a_r} \in A(\mathbb{P}^r)$ is the *multi-indexed exponential generating function*

$$\begin{aligned} \Phi(\mathbf{x}) &:= \sum_{\mathbf{a}} \frac{\mathbf{x}^{\mathbf{a}}}{\mathbf{a}!} I(\mathbf{h}^{\mathbf{a}}) \\ &= \sum_{a_0, \ldots, a_r} \frac{x_0^{a_0} \cdots x_r^{a_r}}{a_0! \cdots a_r!} I\left(\left(h^0\right)^{\bullet a_0} \left(h^1\right)^{\bullet a_1} \cdots (h^r)^{\bullet a_r}\right). \end{aligned}$$

Definition 5.5. The **partial derivatives** of Φ are the usual formal partials of a series:

$$\Phi_{ijk} = \sum_{\mathbf{a}} \frac{\mathbf{x}^{\mathbf{a}}}{\mathbf{a}!} I\left(\mathbf{h}^{\mathbf{a}} \cdot h^i \cdot h^i \cdot h^k\right).$$

Definition 5.6. The **quantum product** for $A^{\bullet}(\mathbb{P}^r)$ is defined by

$$h^i * h^j := \sum_{0 \leqslant k \leqslant r} \Phi_{ijk} h^{r-k}.$$

Slogan. Φ_{ijk} are the structural constants for the quantum product.

Note that the quantum product now has extra parameters $\mathbf{x}^{\mathbf{a}}$ in the Φ_{ijk}.

Definition 5.7. The (big) **quantum cohomology ring** for $\mathbb{P}^r$ is $A^\bullet(\mathbb{P}^r) \otimes_{\mathbb{Z}} \mathbb{Q}[[x]]$. The product structure is obtained by extending the quantum product $\mathbb{Q}[[x]]$-linearly.

The quantum product is obviously commutative. Associativity will be established later.

Example 5.8. (Recovering the usual intersection product.) If one of the three indices of Φ_{ijk} is zero, say $i = 0$, then

$$\begin{aligned}
\Phi_{0jk} &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{d \geqslant 0} I_d \left(\gamma^{\bullet n} \cdot h^0 \cdot h^j \cdot h^k \right) \\
&= I_0 \left(h^0 \cdot h^j \cdot h^k \right) \\
&= \int_{\mathbb{P}^r} h^j \cup h^k.
\end{aligned}$$

Example 5.9. The fundamental class h^0 is the identity for $*$.

Theorem 5.10. *The quantum product $*$ is* ***associative****.*

Proof

(Sketch) We want to show

$$\left(h^i * h^j \right) * h^k = h^i * \left(h^j * h^k \right).$$

Expanding, this is equivalent to

$$\sum_{e+f=r} \sum_{l+m=r} \Phi_{ije} \Phi_{fkl} h^m = \sum_{e+f=r} \sum_{l+m=r} \Phi_{jke} \Phi_{fil} h^m.$$

Since the h^i are independent, we just need to equate coefficients in front of each degree:

$$\sum_{e+f=r} \Phi_{ije}\Phi_{fkl} = \sum_{e+f=r} \Phi_{jke}\Phi_{fil}.$$

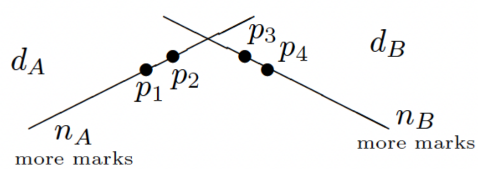
These are the **WDVV equations** (Witten-Dijkgraaf-Verlinde-Verlinde). Further expanding (abbreviating irrelevant classes to $\underline{\gamma}$), we get

$$\begin{aligned}
&\sum_{e+f=r}\sum_{n_A+n_B=n} \frac{n!}{n_A!n_B!} I\left(\gamma^{\bullet n_A}\cdot h^i\cdot h^j\cdot h^e\right) I\left(\gamma^{\bullet n_B}\cdot h^f\cdot h^k\cdot h^l\right)\\
=&\sum_{e+f=r}\sum_{n_A+n_B=n} \frac{n!}{n_A!n_B!} I\left(\gamma^{\bullet n_A}\cdot h^j\cdot h^k\cdot h^e\right) I\left(\gamma^{\bullet n_B}\cdot h^f\cdot h^i\cdot h^l\right).
\end{aligned}$$

This again follows from the **fundamental relation**

$$D\left(p_1p_2 \mid p_3p_4\right) \equiv D\left(p_2p_3 \mid p_1p_4\right)$$

on the moduli space $\overline{M}_{0,n+4}(\mathbb{P}^r, d)$, via the splitting lemma:



■

5.3 Kontsevich's Formula via Associativity

The only non-zero invariants for I_0 are the 3-pointed invariants $I_0(h^i \cdot h^j \cdot h^k) = \int_{\mathbb{P}^r} h^i \cup h^j \cup h^k$. The curve-counting information is contained in the $d > 0$ part.

Let $I_+ := \sum_{d>0} I_d$. Then, we can decompose the full Gromov-Witten potential into $d = 0$ and $d > 0$ parts.

Definition 5.11.

$$\Phi = \Phi^{\text{cl}} + \Gamma$$

where

$$\Phi^{\text{cl}} = \sum_{i,j,k} \frac{x_i x_j x_k}{3!} I_0 \left(h^i \cdot h^j \cdot h^k\right)$$

is called the **classical potential**, and

$$\Gamma = \sum_{n=0}^{\infty} \frac{1}{n!} I_+ \left(\gamma^{\bullet n}\right)$$

is the **quantum potential**.

Similarly, the quantum product decomposes into a classical and a quantum part.

$$\begin{aligned} h^i * h^j &= \sum_{0 \leqslant k \leqslant r} \left(I_0 \left(h^i \cdot h^j \cdot h^k\right) + \Gamma_{ijk}\right) h^{r-k} \\ &= h^i \cup h^j + \sum_{0 \leqslant k \leqslant r} \Gamma_{ijk} h^{r-k}. \end{aligned}$$

(Holds for more general X.)

Slogan. The quantum product is a **deformation** of the classical intersection product.

Theorem 5.12. *Associativity of the quantum product on $\mathbb{P}^2$ implies Kontsevich's recursive formula.*

Proof

For $\mathbb{P}^2$, we have

$$\begin{aligned} h^1 * h^1 &= h^2 + \quad \Gamma_{111}h^1 + \Gamma_{112}h^0 \\ h^1 * h^2 &= \qquad\quad \Gamma_{121}h^1 + \Gamma_{122}h^0 \\ h^2 * h^2 &= \qquad\quad \Gamma_{221}h^1 + \Gamma_{222}h^0. \end{aligned}$$

The associativity

$$\left(h^1 * h^1\right) * h^2 = h^1 * \left(h^1 * h^2\right)$$

expands as

$$\Gamma_{221}h^1 + \Gamma_{222}h^0 + \Gamma_{111}\left(\Gamma_{121}h^1 + \Gamma_{122}h^0\right) + \Gamma_{112}h^2 = \Gamma_{121}\left(h^2 + \Gamma_{111}h^1 + \Gamma_{112}h^0\right) + \Gamma_{122}h^1.$$

Equating coefficients of h^0, we obtain

$$\Gamma_{222} + \Gamma_{111}\Gamma_{122} = \Gamma_{112}\Gamma_{112}.$$

Expanding,

$$\begin{aligned} I_+\left(\left(h^2\right)^{\bullet n} h^2h^2h^2\right) + \sum_{n_A+n_B=n} \frac{n!}{n_A!n_B!} I_+\left(\left(h^2\right)^{\bullet n_A} h^1h^1h^1\right) I_+\left(\left(h^2\right)^{\bullet n_B} h^1h^2h^2\right) \\ = \sum_{n_A+n_B=n} \frac{n!}{n_A!n_B!} I_+\left(\left(h^2\right)^{\bullet n_A} h^1h^1h^2\right) I_+\left(\left(h^2\right)^{\bullet n_B} h^1h^1h^2\right). \end{aligned}$$

Interpreting each I_+ term enumeratively, this is nothing else but Kontsevich's formula!

$$\begin{aligned} N_d + \sum_{d_A+d_B=d} \frac{(3d-4)!}{(3d_A-1)!\,(3d_B-3)!} d_A^3 N_{d_A} d_B N_{d_B} \\ = \sum_{d_A+d_B=d} \frac{(3d-4)!}{(3d_A-2)!\,(3d_B-2)!} d_A^2 N_{d_A} d_B^2 N_{d_B}. \end{aligned}$$

■

5.4 The Small Quantum Cohomology, Algebraic Combinatorics, and More

$\rightsquigarrow$ truncated ring with only the *three-pointed* invariants as structural constants.

$$\begin{aligned}\Phi_{ijk} &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{d>0} d^n \cdot I_d \left(h^i \cdot h^j \cdot h^k\right) \\ &= I_0 \left(h^i \cdot h^j \cdot h^k\right) + q \cdot I_1 \left(h^i \cdot h^j \cdot h^k\right) \quad \text{for } q := \exp(x).\end{aligned}$$

$\rightsquigarrow$ the *original* definition of the quantum cohomology.

$\rightsquigarrow$ small quantum product:

$$h^i * h^j = \begin{cases} h^{i+j} & \text{for } i+j \leqslant r \\ qh^{i+j-r-1} & \text{for } r < i+j \leqslant 2r \end{cases}$$

$\rightsquigarrow$ small quantum cohomology for $\mathbb{P}^r$:

$$\mathbb{Z}[h,q]/\left(h^{r+1} - q\right) \qquad \text{“\textit{deformation of the cohomology ring by the q parameter.}”}$$

$\rightsquigarrow$ “quantum Schubert calculus”

Classically:

Example 5.13. (Pieri rule)

$$(\sigma_b \cdot \sigma_a) = \sum_{\substack{|c|=|a|+b \\ a_i \leqslant c_i \leqslant a_{i-1} \forall i}} \sigma_c$$

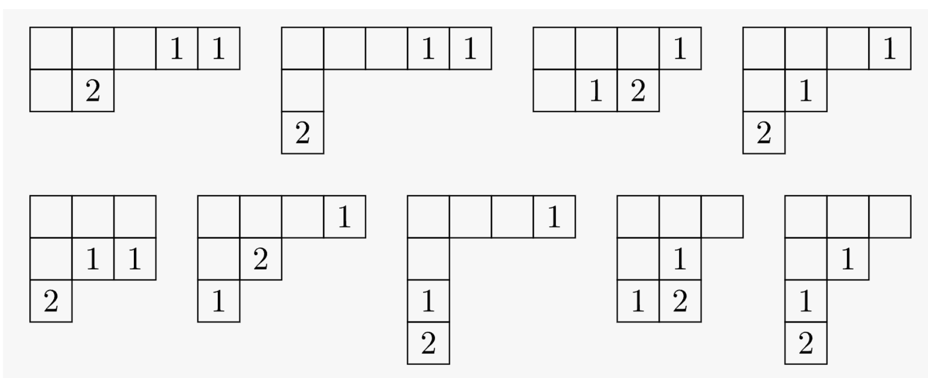
$$\begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 1 & 2 & 2 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 2 \\ \hline 2 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 2 & 2 \\ \cline{1-2} \end{array}$$

Example 5.14. (**Giambelli's formula**)

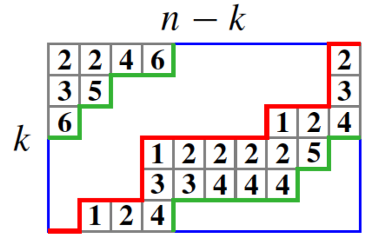
$$\sigma_{a_1,a_2,\ldots,a_k} = \begin{vmatrix} \sigma_{a_1} & \sigma_{a_1+1} & \sigma_{a_1+2} & \cdots & \sigma_{a_1+k-1} \\ \sigma_{a_2-1} & \sigma_{a_2} & \sigma_{a_2+1} & \cdots & \sigma_{a_2+k-2} \\ \sigma_{a_3-2} & \sigma_{a_3-1} & \sigma_{a_3} & \cdots & \sigma_{a_3+k-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sigma_{a_k-k+1} & \sigma_{a_k-k+2} & \sigma_{a_k-k+3} & \cdots & \sigma_{a_k} \end{vmatrix}$$

Example 5.15. (**Littlewood-Richardson coefficients**) The structural constants $c^{\nu}_{\lambda\mu}$ equals the number of skew semistandard Young tableaux of shape λ/μ and filling ν whose reverse row word is ballot.

$$s_{31}s_{21} = s_{52} + s_{511} + s_{43} + 2s_{421} + s_{331} + s_{4111} + s_{322} + s_{3211}$$



⇝ Postnikov (2003): Defined *toric SSYT* and *toric Schur polynomials* and formulated quantum Pieri rule and quantum Kostka numbers in terms of those. Quantum Giambelli formula is the same as the classical formula.



$k = 6,\ n = 16,$
$\lambda[d] = (9, 7, 6, 2, 2, 0)[2],$
$\mu = (9, 9, 7, 3, 3, 1),$
$\beta = (3, 9, 4, 6, 2, 2)$

Figure 5. A semistandard toric tableau of shape $\lambda/d/\mu$

The quantum Littlewood-Richardson rule for tableaux is still wide open till this date.

⇝ Knutson-Tao honeycomb puzzles for equivariant quantum cohomology.

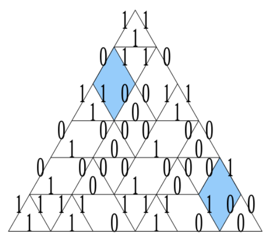


FIGURE 4. A puzzle with two equivariant pieces, which are shaded. The left equivariant piece has weight $y_4 - y_1$, the right $y_5 - y_4$, so this puzzle contributes $(y_4 - y_1)(y_5 - y_4)$ to the calculation of $c_{100101,101010}^{110100}$.

Figure 5.16: An equivariant Knutson-Tao puzzle in the original paper.

⇝ Partial flag varieties, other types (B, C, ...)

⇝ Combinatorics for quantum K-theory.

⇝ Jim Bryan's work on *Banana manifolds* and modular forms.

Useful references:

- (Bertiger 2018) Equivariant Quantum Cohomology of the Grassmannian via the Rim Hook Rule. https://arxiv.org/abs/1403.6218
- (Mihalcea 2004) Equivariant Quantum Schubert Calculus. https://arxiv.org/abs/math/0406066
- (Postnikov 2002) Affine Approach to Quantum Schubert Calculus. https://arxiv.org/abs/math/0205165

- (Bertiger 2022) An Equivariant Quantum Pieri Rule for the Grassmannian on Cylindric Shapes. https://arxiv.org/abs/2010.15395
- (Bryan and Pietromonaco 2024) The Enumerative Geometry and Arithmetic of Banana Nano-Manifolds. https://arxiv.org/abs/2405.04701

Thank you for your attention!