

A LEISURELY INTRODUCTION TO ALGEBRAIC K-THEORY

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ABSTRACT. This write-up is the accompaniment of a talk given in the UW Student AG Seminar in Autumn 2023, and the topic is algebraic K-theory.

Vaguely, K-theory is the universal machine of additive invariants which allows us to study assembly problems in the form of short exact sequences. It originated from the study of assembling vector bundles, now referred to as topological K-theory. The famous theorem(s) due to Serre and Swan establishes a link between vector bundles and finitely generated projective modules. This allows us to transfer the machinery into the land of algebraic K-theory, where we try to assemble projective modules. More recent developments generalizes K-theory to other categories, where these assembly problems receive a more combinatorial treatment.

In this surface-level overview, we begin with a review of topological K-theory. With that, we motivate the constructions of K^0 , K^1 , K^2 in algebraic K-theory, and carry out explicit computations. Then, we move on to discuss classifying spaces and Quillen's constructions for higher K-groups. Finally, we take an excursion on ways to generalize the underlying framework of K-theory to different settings.

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1. K-THEORY ORIGINS: TOPOLOGICAL K-THEORY

First studied by Michael Atiyah and Friedrich Hirzebruch, topological K-theory originates from the study of vector bundles on a topological space. It has produced major classical results in terms of parallelizability of spheres and the existence of division algebras over the reals (the solution of the Hopf invariant one problem due to Frank Adams).

1.1. Vector Bundles. A vector bundle is a family of vector spaces parametrized over some base space.

Definition 1.1. A **vector bundle** E on a base space B is a topological space together with a map $p : E \rightarrow B$ such that for all $b \in B$,

- $p^{-1}(b) \cong F$ for some vector space F (independent of b), and
- **(Local trivialization)** for all $b \in B$, there is a neighborhood U of b and an integer k (the rank) with a homeomorphism $\phi_b : U \times \mathbb{R}^k \rightarrow p^{-1}(U)$ such that the following diagram commutes,

$$\begin{array}{ccc} U \times \mathbb{R}^k & \xrightarrow{\phi_b} & p^{-1}(U) \\ & \searrow & \swarrow \\ & U & \end{array}$$

and the restriction of ϕ_b to each fiber is a linear homomorphism.

Example 1.2. The trivial bundle $X \times \mathbb{R}^n$, the tangent bundle TM , and the normal bundle of a manifold M are all examples of vector bundles.

Vector bundles over a manifold can be reconstructed solely from the data of an atlas and the **transition functions**:

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow GL_k(\mathbb{R}).$$

They satisfy three things

- $g_{\alpha\alpha} = \text{id}$
- $g_{\alpha\beta}(x) = g_{\beta\alpha}(x)^{-1}$
- $g_{\alpha\beta}(x)g_{\beta\gamma}(x)g_{\gamma\alpha}(x) = \text{id}$ for $x \in U_\alpha \cap U_\beta \cap U_\gamma$.

The reconstructed vector bundle would then be

$$E = \coprod_{\alpha} U_{\alpha} \times \mathbb{R}^k / (x, v) \sim (x, g_{\alpha\beta}(x) \cdot v)$$

Fix the base space X and field k . k -vector bundles over X form a category, with objects

$$\begin{array}{c} E \\ \downarrow p \\ B \end{array}$$

and morphisms as commuting triangles

$$\begin{array}{ccc} E & \xrightarrow{\phi} & E' \\ & \searrow & \swarrow \\ & B & \end{array}$$

Let $\text{Vect}_k(B)$ denote the set of isomorphism classes of rank k vector bundles over the base B , and let Gr_k be the infinite Grassmannian, viewed as a colimit $\varinjlim_n \text{Gr}_k(\mathbb{R}^n)$ via the submatrix embedding. There is a bijective correspondence between rank n vector bundles over B and the homotopy class of maps from B to the infinite Grassmannian.

Theorem 1.3 ([5] 1.43). *There is a bijection of sets $\text{Vect}_k(B) \cong [B, \text{Gr}_k]$.*

In other words, *vector bundles up to isomorphism are entirely determined by homotopical data of the base space.*

Remark 1.4. The above result generalizes to any group G and principle G -bundles over B . A principle G -bundle is a fiber bundle with a transitive free G -action on each fiber (making fibers into G -torsors). The transition functions are then elements of G . In that case, isomorphism classes of principle G -bundles bijectively correspond with homotopy classes of maps from B into BG , the **classifying space** of G .

$$\text{Prin}_G(B) \cong [B, BG]$$

1.2. Grothendieck Group and K^0 . There are natural constructions on the set of vector bundles over a fixed base space B .

Definition 1.5. The **Whitney sum** of two vector bundles $E \rightarrow B$ and $E' \rightarrow B$ is formed by direct summing on each fiber and is denoted as $E \oplus E'$.

Definition 1.6. The tensor product of two vector bundles $E \rightarrow B$ and $E' \rightarrow B$ is formed by tensoring on each fiber and is denoted as $E \otimes E'$.

Let $\text{Vect}(B)$ denote isomorphism classes of vector bundles over some fixed B . $\text{Vect}(B)$ naturally forms a **semiring** with these operations — that is, the $\oplus$ and $\otimes$ satisfy all ring axioms *but the existence of additive inverses*. A quick example would be the natural numbers $\mathbb{N}$, missing all the negative integers.

Let us recall what we all did in our first real analysis course. How did we construct the integers? Well, we first assumed mathematical induction and got the natural numbers $\mathbb{N}$. then, **group completion** was the trick we used to get all integers $\mathbb{Z}$. In fact, this construction holds in much more generality including our example. The minimum requirement for the trick to work is having a **commutative monoid**.

Theorem 1.7. Let $U : \text{Ab} \rightarrow \text{CMon}$ be the forgetful functor from the category of Abelian groups to commutative monoids. Then, U has a left adjoint $F : \text{CMon} \rightarrow \text{Ab}$ called the **group completion** functor.

That is to say, every commutative monoid admits a group completion. A common model of such a completion is the **Grothendieck group**.

Definition 1.8. Let A be a semiring (thus a commutative monoid). For pairs of elements $(a, b) \in A \times A$, define the equivalence relation where $(a_1, b_1) \sim (a_2, b_2)$ iff $a_1 + b_2 = a_2 + b_1$. The Grothendieck group of A is the Abelian group

$$\text{Gr}(A) := A \times A / \sim$$

Remark 1.9. The inherited semiring structure makes $\text{Gr}(A)$ a ring.

Example 1.10. $(\mathbb{N}, +, \cdot)$ is a semiring. The pairs of natural numbers $(2, 5) \sim (4, 7)$ because $2 + 7 = 5 + 4$. The Grothendieck group $\text{Gr}(\mathbb{N})$ is $(\mathbb{Z}, +, \cdot)$, where the equivalence class $[(2, 5)]$ is the negative integer -3 .

Now, since $\text{Vect}(B)$ is a semiring, we can do the group completion and get the definition of K^0 .

Definition 1.11. The 0-th K-group of a compact Hausdorff space X is the Grothendieck group of the semiring $\text{Vect}(X)$:

$$K^0(X) := \text{Gr}(\text{Vect}(X))$$

Note that the above definition is over $\mathbb{C}$. We use KO^0 to denote the real K-theory for the same construction over real vector bundles.

1.3. Reduced K-theory $\tilde{K}$. Furthermore, we consider the isomorphism classes of stable vector bundles.

Definition 1.12. Let ε_i denote the trivial rank- i bundle $B \times k^i \rightarrow B$. Two vector bundles $E, E' \rightarrow B$ are **stably isomorphic** if there exists i, j such that $E \oplus \varepsilon_i \cong E' \oplus \varepsilon_j$.

Intuitively, we are killing all classes of trivial bundles in the ring, because what's interesting are the non-trivial bundles. The reduced K-theory is defined to be the K-theory of stably isomorphic vector bundles.

Definition 1.13. The reduced K^0 -group of a topological space X is defined to be

$$\tilde{K}^0(X) := \text{Gr}(\text{Vect}(X) / \sim)$$

Remark 1.14. We will see that K-theories can be made into generalized cohomology theories. In that sense, this definition of reduced K-theory agrees with the **reduced cohomology theory** of K-theory. More explicitly, we have the universal map $\iota : \mathbb{Z} \rightarrow K^0(X)$ by sending $n \mapsto n\varepsilon_1$. Then, $\tilde{K}^0(X) = K^0(X) / \iota(\mathbb{Z})$ where trivial bundles which are all generated by ε_1 are killed.

Remark 1.15. Both $K^0(-)$ and $\tilde{K}^0(-)$ are functorial.

1.4. Higher K-Groups and Bott Periodicity. One definition of K^1 came out first, and it concerns the *automorphism classes* of vector bundles. Nevertheless, the framework of generalized cohomology theories motivates us to make the following generalization:

Definition 1.16. For a space X , let ΣX denote reduced suspension of X . The (reduced) negative K-groups are the $\tilde{K}^0$ of the n -fold suspensions of X .

$$\tilde{K}^{-n}(X) := \tilde{K}^0(\Sigma^n X)$$

The negative indexing here is due to contravariance, but we will later see that this definition carries to positive indices as well. We have an equivalent definition, but it invokes a little category theory:

Theorem 1.17. *There is an adjunction within the category of pointed topological spaces, given by the suspension functor Σ and the loop space functor Ω*

$$\mathrm{Hom}(\Sigma X, Y) \cong \mathrm{Hom}(X, \Omega Y)$$

Now, recall that $\mathrm{Vect}_{\mathbb{C}}(X) \cong [X, \mathrm{BU}]$. We obtain the following by exchanging suspension and loop space:

Proposition 1.18. *The negative K-groups are defined equivalently as*

$$\tilde{K}^{-n}(X) = \mathrm{Gr}([X, \Omega^n \mathrm{BU}])$$

Also, it is good to have an unreduced version defined in the usual way:

Definition 1.19. Let $X_+ = X \sqcup *$. The (unreduced) K-groups of a space X are

$$K^{-n}(X) := \tilde{K}^{-n}(X_+)$$

Now we can get to appreciate the power of topological K-theory. In the 1950s, Raoul Bott discovered arguably the most central result in topological K-theory, a periodicity phenomenon in the homotopy groups of classical Lie groups.

Lemma 1.20. *Lemma 4.6. There are weak equivalences*

$$\Phi : \mathrm{BU} \rightarrow \Omega \mathrm{U}$$

and

$$\begin{aligned} \Phi_1 : \mathrm{BSp} &\rightarrow \Omega(\mathrm{U}/\mathrm{Sp}), & \Phi_2 : \mathrm{BO} &\rightarrow \Omega(\mathrm{U}/\mathrm{O}) \\ \Phi_3 : \mathrm{U}/\mathrm{Sp} &\rightarrow \Omega(\mathrm{SO}/\mathrm{U}), & \Phi_4 : \mathrm{U}/\mathrm{O} &\rightarrow \Omega(\mathrm{Sp}/\mathrm{U}), \\ \Phi_5 : \mathrm{SO}/\mathrm{U} &\rightarrow \Omega \mathrm{SO}, & \Phi_6 : \mathrm{Sp}/\mathrm{U} &\rightarrow \Omega \mathrm{Sp}, \end{aligned}$$

called Bott maps.

Theorem 1.21 ([5] 4.5). *Theorem 4.5 (Bott Periodicity).*

$$\pi_i \mathbf{U} \cong \pi_{i+2} \mathbf{U}$$

$$\pi_i \mathbf{O} \cong \pi_{i+8} \mathbf{O}$$

Corollary 1.22. The K and KO-groups are periodic:

$$\tilde{K}^\bullet(X) \cong \tilde{K}^{\bullet+2}(X)$$

$$\tilde{KO}^\bullet(X) \cong \tilde{KO}^{\bullet+8}(X)$$

Using Bott periodicity, we can finally extend the definition of negative K-groups to all of $\mathbb{Z}$, but its magic does not stop here.

Remark 1.23. We have seen that K^0 classifies vector bundles. What are the higher K-groups measuring? The answer turns out to be automorphism groups of vector bundles.

Consider K^1 as an example. We have

$$K^1(X) \cong [\Sigma X, \mathbf{BU}] \cong [X, \Omega \mathbf{BU}]$$

so each a loop in $\mathbf{BU}$ is assigned to each point of X . The key realization is that each loop in $\mathbf{BU}$ correspond to an automorphism of the vector bundle corresponding to the end point of the loop in $\mathbf{BU}$. Therefore, K^1 captures partial information of automorphism groups of vector bundles.

Remark 1.24. There is an operator-theoretic way of interpreting the K-groups, via the Atiyah-Jänich theorem:

Theorem 1.25. (*Atiyah-Jänich*) Let $\mathcal{H}$ be any separable Hilbert space and $\mathcal{F}(\mathcal{H})$ be the space of Fredholm operators on $\mathcal{H}$. Then, there is a bijective correspondence

$$[X, \mathcal{F}(\mathcal{H})] \cong K^0(X)$$

This correspondence leads to developments in what's called operator K-theory.

1.5. Applications of Topological K-Theory. The discovery of Bott periodicity led to a few major breakthroughs in algebraic topology. In specific, it solves the famous Hopf invariant one problem concerning the existence of normed division algebras over $\mathbb{R}$. Briefly, the Hopf invariant is a number associated with maps $f : S^{2n-1} \rightarrow S^{2n}$ which equals 1 only if there is a corresponding

Hopf fibration. Adams proved first using the Steenrod algebra and then K-theory that this can only happen when $n = 1, 2, 4$.

Theorem 1.26 ([2]). *There exists normed division algebras over $\mathbb{R}$ only in dimensions 1, 2, 4, and 8. These are $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$ upto isomorphism. Correspondingly, the only parallelizable spheres are S^1 , S^3 , and S^7 .*

A nice schematic Figure 1.27 is provided in Adam's original paper [[11] p60].

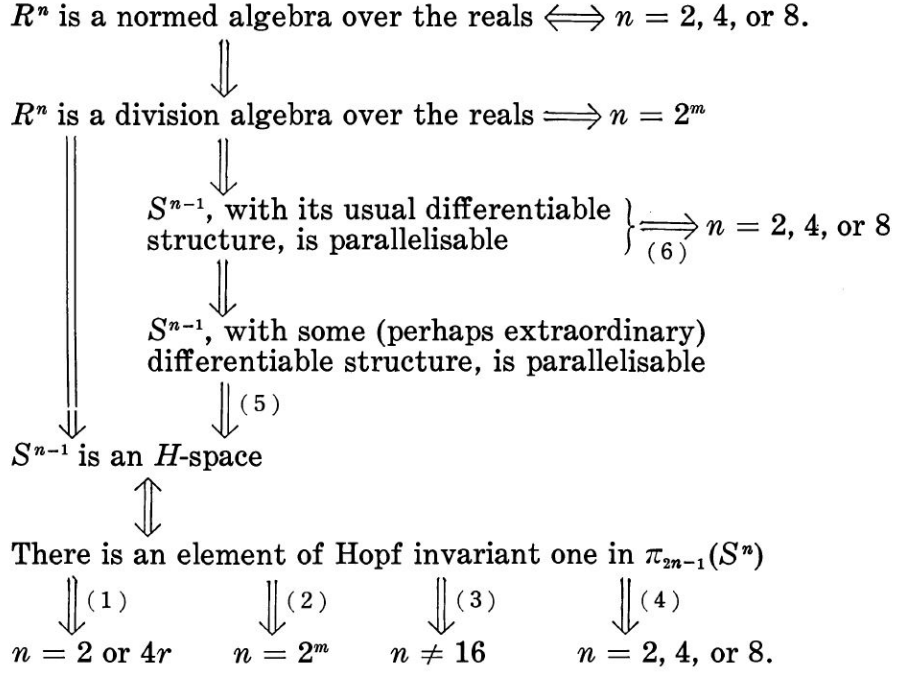


FIGURE 1.27. Adam's Flow Chart

Using K-Theory, Adams also gave a sharp bound on the maximum number of everywhere linearly independent vector fields on spheres.

Definition 1.28. For n written as the product of an odd number A and a power of two 2^B , write

$$B = c + 4d, 0 \leq c < 4.$$

Then, the Radon-Hurwitz number is defined as

$$\rho(n) = 2^c + 8d.$$

Theorem 1.29 ([3] p1036). *The maximum number of everywhere linearly independent vector fields on S^n is $\rho(n) - 1$.*

1.6. Generalized Cohomology Theory. K-Theory is a generalized cohomology theory in that it satisfies all the Eilenberg-Steenrod axioms but the dimension axiom. A **spectrum** is a sequence of spaces $E = \{E_i\}$ with structure maps $\Sigma E_i \rightarrow E_{i+1}$ (dually $E_i \rightarrow \Omega E_{i+1}$). *Brown Representability Theorem* says that every generalized cohomology theory is represented by a spectrum.

Theorem 1.30. *(Complex) K-theory is represented by the K-theory spectrum KU in the sense that $K(X) \cong [X, KU]$, where*

$$KU_i = \begin{cases} \mathbb{Z} \times BU & i \text{ even} \\ U & i \text{ odd.} \end{cases}$$

Similarly, KO-theory is represented by the spectrum KO .

2. FROM TOPOLOGY TO ALGEBRA: THE SERRE-SWAN THEOREM

What does all of that has to do with algebraic geometry? The answer is that vector bundles should really be thought of as projective modules over some coordinate ring.

First, recall the following equivalent definitions of projective (right) modules:

Proposition 2.1. *P is a projective R-module iff*

- *P is a direct summand of a free module.*
- *P has the lifting property.*
- *$\text{Hom}(P, -)$ is exact.*
- *Every SES ending in P splits.*

The following lemma immediately suggests a close connection between bundles and projective modules.

Lemma 2.2. *Every finite rank vector bundle η over a compact Hausdorff space X is a direct summand of a trivial bundle.*

Proof. Since X is compact Hausdorff, it is covered by finitely many opens U_i . Choose s_i^j from global sections to form a local basis over each U_i . Again, there are finitely many of them, say the number is N . Hence, we can form the trivial bundle of rank N ,

$$\xi : X \times \mathbb{R}^N \rightarrow X$$

with generators e_i^j indexed in the same way, and a surjection

$$\pi : \xi \rightarrow \eta, e_i^j \mapsto s_i^j$$

We then have a SES

$$0 \rightarrow \ker \pi \rightarrow \xi \rightarrow \eta \rightarrow 0$$

which splits by the orthogonal projection $\xi \rightarrow \ker \pi$. Therefore,

$$\eta \oplus \ker \pi \cong \xi$$

□

The above fact results in the correspondence between vector bundles and projective modules in a number of settings. First, there is **Swan's theorem** which deals with *topological vector bundles*.

Theorem 2.3 (Swan '62). *Given a compact Hausdorff space X , the category of finitely generated projective modules $\text{Proj}(C(X))$ over the continuous-function algebra $C(X)$ is equivalent to the category of finite-rank vector bundles $\text{Vect}(X)$.*

Proof. (Sketch) The direction

$$\text{Vect}(X) \rightarrow \text{Proj}(C(X))$$

is clearly given by the global sections functor:

$$(\eta : E \rightarrow X) \mapsto \Gamma(\eta, C(X))$$

By Theorem 2.2, $\eta \oplus \zeta \cong \xi$ for some trivial bundle ξ , and $\Gamma(-, C(X))$ takes a split mono to a split mono, so we obtain

$$\Gamma(\eta, C(X)) \oplus \Gamma(\zeta, C(X)) \cong \Gamma(\xi, C(X))$$

where $\Gamma(\xi, X)$ is finitely generated free since ξ is trivial of finite rank. We therefore conclude that $\Gamma(\eta, C(X))$ is projective.

For the direction of

$$\text{Proj}(C(X)) \rightarrow \text{Vect}(X),$$

take any projective module P , being a direct summand of free: $P \oplus M \cong C(X)^n =: F$. This means that there is an idempotent $\text{hom } f : F \rightarrow F$ with image P . F is given by any trivial rank- n bundle, so say $F = \Gamma(\xi, C(X))$. Then, f corresponds to some map $\varphi : \xi \rightarrow \xi$ of bundles such that $f = \varphi$. After a little work checking everything is well-defined, we see that P corresponds to the image of φ , which is a subbundle of ξ .

□

We can transfer this result to smooth bundles and get the **smooth Serre-Swan theorem**:

Theorem 2.4. *Given a differentiable manifold M , the category of finitely generated projective modules $\text{Proj}(C^\infty(M))$ over the algebra of smooth functions $C^\infty(M)$ is equivalent to the category of finite-rank vector bundles $\text{Vect}(M)$.*

The original **Serre's theorem** concerns more or less with **algebraic vector bundles** over affine varieties. It was used to answer the question “*Is every projective module over $k[x_1, \dots, x_n]$ free?*” and the answer was YES.

Theorem 2.5 (Serre 55). *let R be commutative unital Noetherian (in particular, the coordinate ring of an affine variety over a field). The category of **finitely-generated projective** R modules is equivalent to the category of algebraic vector bundles — **locally free** sheaves of structure sheaf-modules of constant finite rank) on $\text{Spec} R$.*

Indeed, a more general phenomenon is taking place in the background. As discussed in the notes by Archana S. Morye in [7], the Serre-Swan correspondence holds in the generality of *locally ringed spaces*.

For any ringed space $(X, \mathcal{O}_X)$, let $\mathcal{O}_X\mathbf{Mod}$ denote the category of $\mathcal{O}_X$ -modules, and $\mathbf{Lfb}(X)$ the full subcategory of $\mathcal{O}_X\text{-mod}$ consisting of locally free $\mathcal{O}_X$ -modules of bounded rank. For any ring A , let $A\text{-mod}$ denote the category of A -modules, and $\mathbf{Fgp}(A)$ the full subcategory of $A\text{-mod}$ consisting of finitely generated projective A -modules.

Theorem 2.6 ([7] Theorem 2.1). *Let $(X, \mathcal{O}_X)$ be a locally ringed space, and let $A = \Gamma(X, \mathcal{O}_X)$. Assume that $\mathcal{O}_X\text{-mod}$ contains a certain nice subcategory, and that every sheaf in $\mathbf{Lfb}(X)$ is finitely generated by global sections. Then, $\Gamma(X, \bullet) : \mathbf{Lfb}(X) \rightarrow \mathbf{Fgp}(A)$ is an equivalence of categories, i.e., the Serre-Swan Theorem holds for $(X, \mathcal{O}_X)$.*

3. ALGEBRAIC K-THEORY: CLASSICAL CONSTRUCTIONS

We have seen the analogy between vector bundles and projective modules in the Serre-Swan theorem. Up until this section, we have been working with modules over the ring of continuous sections, but there are so many other base rings that we can choose from! Transferring the machinery of K-theory into the land of algebraic geometry, we hope to be able to cook up some invariants which give us useful information about the classification of projective modules.

Throughout this chapter, the base ring R is assumed to be *associative unital* unless otherwise specified (to be commutative).

3.1. The Grothendieck Group K_0 . For R *associative unital*, let $\text{Proj}(R)$ denote the isomorphism classes of finitely generated projective R -modules. $\text{Proj}(R)$ is naturally endowed with direct sum $\oplus$ and tensor $\otimes$, making it a commutative monoid. As a consequence of Serre-Swan, the construction of K^0 transfers immediately to isomorphism classes of projective modules, giving us an invariant that classifies projective modules.

Definition 3.1. The zeroth algebraic K-group of the ring R is

$$K_0 R := \text{Gr}(\text{Proj}(R))$$

Analogously, we have the notion of a reduced K-group, defined via “stably isomorphic” projective modules. Define an equivalence relationship on $\text{Proj}(R)$, where $M \sim N$ whenever there are free modules such that $M \oplus R^n \cong N \oplus R^m$. Then, modulo $\sim$, we have

Definition 3.2. The reduced zeroth algebraic K-group of the ring R is

$$\tilde{K}_0 R := \text{Gr}(\text{Proj}(R)) / \sim$$

Per previous discussion, reduced K-groups ought to be interpreted in terms of reduced generalized cohomology theories, which remains true for algebraic K-theory. Again, use the canonical map $\iota : \mathbb{Z} \rightarrow K_0 R, n \mapsto n[R]$. Then,

$$K_0 R = \tilde{K}_0 R / \iota(\mathbb{Z})$$

where all free modules are killed.

In the case of commutative rings, we have the following exact sequence

$$0 \rightarrow \mathbb{Z} \rightarrow K_0 R \rightarrow \tilde{K}_0 R \rightarrow 0$$

which splits. As a consequence,

Proposition 3.3. *For a commutative unital ring R ,*

$$K_0 R \cong \tilde{K}_0 R \oplus \mathbb{Z}$$

Remark 3.4. Why do we restrict our attention to finitely generated modules? It turns out that something quite silly will happen if we allow modules of infinite rank. Suppose P is a projective modules such that $P \oplus Q \cong R^n$. Then,

$$\begin{aligned} P \oplus R^\infty &= P \oplus R^n \oplus R^n \oplus \dots \\ &= P \oplus (Q \oplus P) \oplus (Q \oplus P) \oplus \dots \\ &= (P \oplus Q) \oplus (P \oplus Q) \oplus \dots \\ &= R^n \oplus R^n \oplus \dots \\ &= R^\infty \end{aligned}$$

so everything becomes stably free and therefore trivial! This is known as the **Eilenberg Swindle**.

3.1.1. *A Tangent on Group Completions.* The reader might have seen another model for group completions.

Proposition 3.5. *The following is an equivalent construction for a group completion. Given a commutative monoid M , let $F(M)$ be the free abelian group on the letters $[m]$ for all $m \in M$. Then, the group completion of M is*

$$\text{Gr}(M) := F(M) / \langle [m+n] - [m] - [n] \rangle$$

We have seen that $\text{Gr}(\mathbb{N}) = \mathbb{Z}$. Now let us go on a tangent and compute some other group completions.

Example 3.6. The non-zero integers $\mathbb{Z}_{\neq 0}$ under multiplication is a commutative monoid. Its group completion is $\mathbb{Q}^\times$. This can be easily seen: in $\text{Gr}(\mathbb{Z}_{\neq 0})$, $(a, b) \sim (c, d)$ iff $ad = bc$, but that is the same equivalence relation which defines $\mathbb{Q}$.

The following two examples concerns the representation theory of finite groups.

Example 3.7 ([1] Example 2.1.5). **Burnside Ring.** Let $\mathbf{GSets}_{\text{fin}}$ be the category (in fact, the set) of isomorphism classes of finite G -sets. $\mathbf{GSets}_{\text{fin}}$ is a commutative monoid under disjoint union and direct product. If X is a finite G -set, then we can decompose it into a disjoing union of orbits,

$$X = \bigsqcup X_i$$

By stabilizer-orbit, we know that each $X_i \cong G/\text{Stab}(x_i)$ for an element $x_i \in X_i$, and all subgroups of G can be $\text{Stab}(x_i)$ for some orbit X_i . Also, if $H, H' \subseteq G$ are conjugate, then $G/H \cong G/H'$. Therefore, the set of all possible non-isomorphic G -orbits is just the set of conjugacy classes of G , which is again a finite set. Suppose that G has c conjugacy classes. Then, $\mathbf{GSets}_{\text{fin}}$ being generated by the c non-isomorphic G -orbits, we have

$$\mathbf{GSets}_{\text{fin}} \cong \mathbb{N}^c$$

whence the group completion of $\mathbf{GSets}_{\text{fin}}$ equals

$$A(G) := \text{Gr}(\mathbf{GSets}_{\text{fin}}) \cong \mathbb{Z}^c$$

With the direct product, $A(G)$ is called the Burnside ring of G . As an example,

$$A(C_p) = \mathbb{Z}[x]/(x^2 - px)$$

Example 3.8 ([1] Example 2.1.6). **Representation ring.** For a finite group G , let $\text{Rep}_{\mathbb{C}}(G)$ be the set of finite dimensional $\mathbb{C}$ -representations of G . $\text{Rep}_{\mathbb{C}}(G)$ is a commutative monoid under $\oplus$ and $\otimes$. By Maschke's theorem, every $\mathbb{C}$ -representation is completely reducible, and all non-isomorphic irreducible representations are one-dimensional and depend only on the conjugacy classes of G . Hence,

$$\text{Rep}_{\mathbb{C}}(G) \cong \mathbb{N}^c$$

again, and its group completion is

$$R_{\mathbb{C}}(G) := \text{Gr}(\text{Rep}_{\mathbb{C}})(G) \cong \mathbb{Z}^c$$

With the tensor product, $R_{\mathbb{C}}(G)$ is called the ($\mathbb{C}$ -) representation ring of G . As an example,

$$R_{\mathbb{C}}(C_p) = \mathbb{Z}[x]/(x^p - 1)$$

and more generally, if M is a finite abelian group, then

$$R_{\mathbb{C}}(M) \cong \mathbb{Z}[\hat{M}]$$

where $\hat{M}$ is the group of $\mathbb{C}$ -characters of M .

3.2. Computing Algebraic K_0 . Algebraic K_0 has been computed in a lot of instances. Let us look at some simplest examples.

Example 3.9. Fields. $R = k$ where k is a field. Then, finitely generated projective modules over k are just finite-dimensional k -vector spaces, so $\text{Proj}(k) \cong \mathbb{N}$ via taking the rank and $K_0(k) \cong \mathbb{Z}$.

Example 3.10. PID. Let R be a PID. We know that finitely generated projective modules over a PID are free, so again $\text{Proj}(R) \cong \mathbb{N}$ via taking the rank, and $K_0(R) \cong \mathbb{Z}$. As a consequence, $K_0(\mathbb{Z}) \cong \mathbb{Z}$.

The definition of algebraic K_0 recovers the topological K^0 groups.

Example 3.11. Continuous functions. Let R be $C(X, \mathbb{R})$, the ring of continuous function from a space X to $\mathbb{R}$. Via Serre-Swan, we see that

$$K_0 C(X, \mathbb{R}) \cong KO^0(X)$$

Similarly,

$$K_0 C(X, \mathbb{C}) \cong K^0(X)$$

Example 3.12 ([4] p2). Dedekind domain. Let R be a Dedekind domain. This means that R has *unique factorization of ideals*, and its *ideal classes* are the equivalence classes of fractional ideals $\sim$: $a \sim b$ iff exists x, y such that $xa = yb$. The ideal classes of R therefore forms a group under multiplication, and it is the **ideal class group** of R , denoted $\text{Cl}(R)$. (One quickly sees that this is equivalent to saying $\text{Cl}(R) := J(R)/P(R)$)

Now, recall that every finitely generated projective R -module is a direct sum of ideals, and vice versa. Therefore, $\text{Proj}(R)$ is isomorphic as a commutative monoid to the free monoid on all fractional ideals of R . After group completion, we have the following isomorphism:

$$K_0(R) \cong \mathbb{Z} \oplus \text{Cl}(R), [a_0 \oplus \dots \oplus a_l] \mapsto (l, [a_1, \dots, a_l])$$

Let us make this concrete via the following classic example of a Dedekind domain. Let $R = \mathbb{Z}[\sqrt{-5}]$. This is a Dedekind domain but not a UFD, because $2 \times 3 = (1 + \sqrt{-5})(1 - \sqrt{-5})$. $\text{Cl}(R) \cong \mathbb{Z}/2$ with generator $(3, 2 + \sqrt{-5})$. Therefore, $K_0(\mathbb{Z}[\sqrt{-5}]) \cong \mathbb{Z} \oplus \mathbb{Z}/2$.

Seen in the following propositions, K_0 behave nicely with other standard constructions in the category of associative unital rings, which gives us more computational tools.

Proposition 3.13. K_0 is functorial.

Proposition 3.14. K_0 commutes with direct products.

$$K_0(R_1 \times R_2) \cong K_0(R_1) \times K_0(R_2)$$

Proposition 3.15 ([1] 2.1.6). K_0 commutes with filtered colimits.

$$K_0(\varinjlim R_i) \cong \varinjlim K_0(R_i)$$

The next few examples deal with cases where all modules are projective. Recall, a ring R is simple (resp. semisimple) if it is a simple (resp. semisimple) module over itself.

Proposition 3.16. TFAE:

- R is semisimple.
- R is semisimple Artinian.
- Any right R -module is projective.
- Any right R -module is injective.
- R has no non-zero nilpotent right ideals.

A famous theorem by Artin and Wedderburn classifies all semisimple rings.

Theorem 3.17 (Wedderburn-Artin). If R is semisimple, then there exists division algebras $D_1, \dots, D_r$ such that R is isomorphic to the direct product of matrix algebras over the D_i .

$$R \cong M_{n_1}(D_1) \times \dots \times M_{n_r}(D_r)$$

Example 3.18 ([1] 2.1.2). **Simple rings.** If R is simple, then it is semisimple, so every R -module is projective. Taking the length is an invariant of finitely generated R -modules, which induces an isomorphism $K_0(R) \cong \mathbb{Z}$. As an example, the matrix algebra $M_n(D)$ is simple for any division

algebra D , so

$$M_n(D) \cong \mathbb{Z}$$

Example 3.19 ([1] 2.1.4). **Semisimple rings.** If R is semisimple, then by Wedderburn-Artin, $R \cong M_{n_1}(D_1) \times \dots \times M_{n_r}(D_r)$. Therefore,

$$K_0(R) \cong K_0(M_{n_1}(D_1)) \times \dots \times K_0(M_{n_r}(D_r)) \cong \mathbb{Z}^r$$

Since K_0 depends only on $\text{Proj}(R)$, it should be invariant when two rings have equivalent categories of modules.

Definition 3.20. Rings R and S are **Morita equivalent** if $\mathbf{RMod}$ and $\mathbf{SMod}$ are equivalent as *abelian categories*.

In particular, an equivalence of abelian categories is a pair of additive functors, say

$$\mathbf{RMod} \begin{array}{c} \xrightarrow{T} \\ \xleftarrow{U} \end{array} \mathbf{SMod}$$

Let $P = T(R)$ and $Q = U(S)$. The following theorem holds.

Theorem 3.21 ([1] Theorem 2.7). **Structure Theorem for Morita Equivalence.** R, S, P, Q as above.

- P and Q are finitely generated projective.
- $\text{End}_S(P) \cong R \cong \text{End}_S(Q)^{\text{op}}$ and $\text{End}_R(Q) \cong S \cong \text{End}_R(P)^{\text{op}}$.
- $P \cong \text{Hom}_R(Q, R)$ and $Q \cong \text{Hom}_S(P, S)$.
- $T(M) \cong M \otimes_R P$ and $U(N) \cong N \otimes_S Q$.
- The functors $\text{Hom} - S(P, -)$ and $\text{Hom}_R(Q, -)$ are faithful.

Corollary 3.22. If R and S are Morita equivalent, then $K_0(R) \cong K_0(S)$.

Example 3.23 ([1] 2.7.2). For any ring R , $M_n(R)$ is always Morita equivalent to R , so

$$K_0(M_n(R)) \cong K_0(R)$$

3.3. The Classical Construction of K_1 . We have seen that K_0 classifies finitely generated projective modules over the ring R . We also recall that higher topological K -groups captured information about automorphism groups of vector bundles. Therefore, it is tempting to cook up higher invariants that tells us information about automorphism groups of projective modules. In other words, we need to understand the general linear group over the ring R .

Definition 3.24. Let R be an *associative unital* ring. The general linear group of rank n over R , denoted $GL_n(R)$, is the group of automorphisms of the free module R^n . In matrix form, $GL_n(R)$ consists of $n \times n$ invertible matrices with coefficients in R .

Remark 3.25. In the case that R is *commutative unital*, the determinant makes sense, and $GL_n(R)$ consists of matrices whose determinant equals a unit in R .

Similar to the classical case, there are canonical inclusions of $GL_n(R) \hookrightarrow GL_{n+1}(R)$ as the upper left block: $A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$. We define the (infinite) general linear group as the colimit of these inclusions.

Definition 3.26. The general linear group of R is

$$GL(R) := \varinjlim GL_n(R)$$

John Milnor in the early 1960s discovered the correct definition for algebraic K_1 , as the abelianization of $GL(R)$.

Definition 3.27. Let $[-, -]$ denote taking commutators. The first algebraic K-group of the ring R is defined as follows.

$$K_1 R := GL(R) / [GL(R), GL(R)]$$

It turns out that this commutator subgroup can be explicitly computed as the subgroup generated by the set of elementary matrices $\{e_{ij}\}$.

Definition 3.28. Let e_{ij} be the elementary matrix whose only none-zero entry is 1 at position (i, j) , for all $i, j \in \mathbb{N}$. Define the subgroup of $GL(R)$,

$$E(R) := \langle re_{ij} \mid r \in R, i, j \in \mathbb{N} \rangle$$

Proposition 3.29.

$$[GL(R), GL(R)] \cong E(R)$$

Also, $E(R)$ is a perfect group because it is equal to its commutator subgroup.

Proposition 3.30.

$$[E(R), E(R)] \cong E(R)$$

There is a familiar face that appears in the colimit $GL(R)$, namely $GL_1(R) \cong R^\times$. Therefore, we have

Proposition 3.31. K_1 always contains the abelianization of the group of units.

$$R_{ab}^\times \subseteq K_1 R$$

Proposition 3.32 ([1] Proposition 1.6.4). For R, S that are Morita equivalent, we have

$$K_1(R) \cong K_1(S)$$

3.4. Computing Algebraic K_1 .

Example 3.33. SK_1 . If R is commutative unital, then $\det : GL(R) \rightarrow R^\times$ is a surjective group hom, and it further induces a surjection $\det : K_1(R) \rightarrow R^\times$. We write $SK_1(R)$ to be the kernel of $K_1(R) \rightarrow R^\times$. Also, observe that the kernel of $GL(R) \rightarrow R^\times$ consists of the matrices with determinant 1, so it is $SL(R)$! Therefore, we have the following SES

$$0 \rightarrow SL(R) \rightarrow GL(R) \rightarrow R^\times \rightarrow 0$$

which splits by Theorem 3.31.

Therefore,

$$GL(R) \cong SL(R) \rtimes R^\times$$

which induces the direct sum decomposition

$$K_1(R) \cong R^\times \oplus SK_1(R)$$

after abelianization.

Example 3.34 ([1] 1.3.5). $M_n(R)$. The Morita equivalence between R and $M_n(R)$ again induces

$$K_1(M_n(R)) \cong K_1(R)$$

Example 3.35. Fields. For k a field, $SL(k) \cong [GL(k), GL(k)]$, which means

$$K_1(k) \cong k^\times$$

Example 3.36 ([1] 1.3.5). **Euclidean domains.** For R a Euclidean domain, it can be show that $SK_1(R) = 0$, whence

$$K_1(R) \cong R^\times$$

As examples, $K_1(\mathbb{Z}) \cong \{\pm 1\}$ and $K_1(k[x]) \cong k^\times$.

3.5. The Classical Construction of K_2 . Beyond K_0 and K_1 , defining higher invariants becomes extremely difficult. A good definition of algebraic K_2 was first given by John Milnor in 1963 and then refined by Hyman Bass in 1967, which involves the use of *Steinberg symbols*.

Definition 3.37. Let $n \geq 3$. The **Steinberg group** $St_n(R)$ is defined via the following generators

$$\{x_{ij}^r \mid r \in R, 1 \leq i \neq j \leq n\}$$

and relations

$$\begin{cases} x_{ij}^r x_{ij}^s = x_{ij}^{r+s} \\ [x_{ij}^r, x_{j\ell}^s] = x_{i\ell}^{rs} & \text{if } i \neq \ell, \\ [x_{ij}^r, x_{k\ell}^s] = 1 & \text{if } i \neq \ell \text{ and } j \neq k \end{cases}$$

As usual, the (infinite) Steinberg group is defined as the colimit of $ST_n(R)$

The astute reader might have immediately noticed the similarity between the indexing of x_{ij}^r and re_{ij} . Indeed, all re_{ij} satisfy the above relations, so the map sending $x_{ij}^r \mapsto re_{ij}$ is a surjective group hom. K_2 is then defined to be the kernel of this homomorphism.

Definition 3.38. Let $\varphi : St(R) \rightarrow E(R)$ be the surjective group hom sending $x_{ij}^r \mapsto re_{ij}$. Then,

$$K_2 R := \ker \varphi$$

We can compose φ with the injection $E(R) \rightarrow GL(R)$ to get $\tilde{\varphi} : St(R) \rightarrow GL(R)$ whose image equals $E(R)$. We further have the quotient map $GL(R) \rightarrow K_1(R)$ whose kernel equals $[GL(R), GL(R)] = E(R)$, so these maps fit into the following six-term exact sequence

$$0 \rightarrow K_2(R) \rightarrow St(R) \rightarrow GL(R) \rightarrow K_1(R) \rightarrow 0$$

making $K_1(R)$ the cokernel of $\tilde{\varphi}$.

Recall, for $0 \rightarrow A \rightarrow E \rightarrow G \rightarrow 0$, E is a *central extension* of G by A if $A \subseteq Z(E)$. Also recall that central extensions are measured by the second group cohomology.

Proposition 3.39. *Isomorphism classes of central extensions of G by A bijectively correspond to $H^2(G, A)$.*

A central extension $C \rightarrow G$ is *universal* if it is the universal object in the category of central extensions $\{E \rightarrow G\}$ with morphisms as triangles. If $\alpha : C \rightarrow G$ is universal, then $\ker \alpha \cong H_2(G, \mathbb{Z})$ canonically.

Theorem 3.40 ([1] III.5.4, Recognition Theorem). *Every perfect group G has a universal central extension*

$$1 \rightarrow H_2(G; \mathbb{Z}) \rightarrow [F, F]/[R, F] \rightarrow G \rightarrow 1$$

where F is some free group surjecting onto G .

Theorem 3.41 ([1] III.5.5 Kervaire, Steinberg). *The surjection $\text{St}(R) \rightarrow E(R)$ is the universal central extension of $E(R)$. As a consequence,*

$$K_2(R) \cong H_2(E(R), \mathbb{Z})$$

3.6. Computing Algebraic K_2 . The Steinberg symbols provide an explicit way of computing algebraic K_2 , but the computation is still hopeless for many cases.

Definition 3.42 ([1] III.5.10). For $A, B \in E(R)$, choose $a \in \varphi^{-1}(A)$ and $b \in \varphi^{-1}(B)$. Then, define

$$A \star B := [a, b] \in K_2(R)$$

Since $K_2(R)$ is central in $\text{St}(R)$, $[a, b] = [a', b']$ for different representatives, so the above is well-defined. Also let $\text{diag}(-)$ denote the diagonal matrix with diagonal entries $(-)$. Then, we are able to construct the Steinberg symbol.

Definition 3.43. Let $r, s \in R^\times$ be commuting elements. The **Steinberg symbol** $\{r, s\}$ is defined as

$$\{r, s\} := \text{diag}(r, r^{-1}, 1) \star \text{diag}(s, 1, s^{-1})$$

The Steinberg symbol satisfies the following relations:

Proposition 3.44. *Whenever defined, the Steinberg symbol satisfies*

- $\{r, s_1 s_2\} = \{r, s_1\} + \{r, s_2\}$
- $\{r_1 r_2, s\} = \{r_1, s\} + \{r_2, s\}$
- (*Steinberg Relation*) $\{r, 1 - r\} = 0$

Let $K_2^M(R)$ be the group generated by $\{r, s\}$ with the above relations. $K_2^M(R)$ is clearly abelian. In fact, we can check that all Steinberg symbols are in the kernel of φ , whence $K_2^M(R) \subseteq K_2(R)$. The famous theorem due to Matsumoto Hideyuki shows that these groups are in fact isomorphic for fields.

Theorem 3.45 (Matsumoto). *For a field k , the homomorphism $K_2^M(R) \rightarrow K_2(R)$ is an isomorphism.*

4. ALGEBRAIC K-THEORY: HIGHER K-GROUPS

K_0, K_1 and K_2 are defined classically purely in terms of algebra. Also, we already know that for certain R (e.g. a Dedekind domain), then these groups fit into "part of" a LES,

$$K_2(R) \rightarrow K_2(\text{Frac}(R)) \rightarrow \bigoplus_{\mathfrak{m}} K_1(R/\mathfrak{m}) \rightarrow K_1(R) \rightarrow K_1(\text{Frac}(R)) \rightarrow \bigoplus_{\mathfrak{m}} K_0(R/\mathfrak{m}) \rightarrow K_0(R) \rightarrow K_0(\text{Frac}(R))$$

so perhaps we should try to extend the definition to higher degrees. However, it turned out that coming up with a reasonable definition for the higher K-groups was a serious challenge. Milnor and others have respectively attempted at the problem in the 70s, but it was not until Daniel Quillen's constructions which borrowed ideas from homotopy theory that a widely recognized definition became available. Later on, Quillen's constructions began to exhibit real power, and people were able to extend the definition of K-theory further to other categories.

4.1. Classifying Spaces of a Small Category. The key ingredient to Quillen's constructions is one that's inspired by algebraic topology, namely *classifying spaces*. It should be thought of as a representing object in the same sense as a moduli space or a parameter space.

Recall, a **small** category is one which can be embedded into **Sets**. In other words, if $\mathcal{C}$ is a small category, then both $\text{ob}(\mathcal{C})$ and each $\text{Hom}_{\mathcal{C}}(A, B)$ are sets.

Also recall, a **simplicial object** in a category $\mathcal{C}$ is a contravariant functor $\Delta^{\text{op}} \rightarrow \mathcal{C}$. Here, Δ is the "universal monoidal category" where objects are $[n]$ and morphisms are order-preserving functions. (Δ should be thought of as the "infinite" simplex). In specific, a **simplicial set** is a contravariant functor $\Delta^{\text{op}} \rightarrow \mathbf{Sets}$. Now, let's try to build a topological space from a small category.

Definition 4.1. The **Nerve** $N(\mathcal{C})$ of a small category $\mathcal{C}$ is a simplicial set, where

$$[n] \mapsto \{\text{n-composable morphisms in } \mathcal{C}\}$$

Every simplicial set has a **geometric realization** obtained by adding an n -simplex Δ^n to every element in the image of $[n]$ and gluing them as a CW-complex according to the face maps and degeneracy maps of $[n]$.

Definition 4.2. The **classifying space** BC of a small category $\mathcal{C}$ is the geometric realization of its nerve $N(\mathcal{C})$. In math notation,

$$BC := |N(\mathcal{C})|$$

Recall, a group is equivalent to a small category where there is only one object and morphisms are group elements. Then, one would naturally be concerned of this new definition, because the classifying space for principal G -bundles has been classically defined as EG/G (recall, EG is the contractible space with a free G -action). Indeed, these two notions agree.

Proposition 4.3.

$$|N(G)| \cong EG/G$$

Let's see some examples.

Example 4.4.

$$B\mathbb{Z} \cong \mathbb{R}$$

$$BC_2 \cong \mathbb{RP}^\infty$$

With S^∞ understood as a subspace of $\mathbb{C}^\infty$ and C_n acting by multiplication with roots of unity,

$$BC_n \cong S^\infty/C_n$$

Finally, as we have seen in section 1,

$$BO \cong Gr_\infty$$

4.2. The Plus Construction and the K-Theory Space KR . We have seen classically that algebraic K-groups deal with subgroups and extension properties of $GL(R)$, so the right thing to do here is to form the classifying space $BGL(R)$. Our eventual goal is to define higher algebraic K-groups as the homotopy groups of some topological space related to $BGL(R)$, and the last missing ingredient is Quillen's **plus construction**.

Definition 4.5 ([1] IV.1.1). The plus construction $BGL(R)^+$ denotes any CW complex X having a distinguished map from $BGL(R) \rightarrow BGL(R)^+$ such that

- $\pi_1 BGL(R)^+ \cong K_1(R)$, and the natural map $GL(R) = \pi_1 BGL(R) \rightarrow \pi_1 BGL(R)^+$ is surjective with kernel $E(R)$.
- $H_\bullet(BGL(R); M) \cong H_\bullet(BGL(R)^+; M)$ for all $M \in K_1(R)\mathbf{Mod}$.

As with many of these definitions, multiple *models* are available, and we mention the following 2:

- Construct $BGL(\mathbb{Z})^+$ by attaching 2 and 3-cells to kill off unwanted homotopy groups, and let $BGL(R)^+$ be the pushout of $BGL(\mathbb{Z})^+$ along the universal map $\mathbb{Z} \rightarrow R$.
- "Group completing" the H-space $BGL(R)$ to get an infinite loop space. The basepoint component will be $BGL(R)^+$.

That was quite a detour, but we are now ready to define the higher algebraic K-groups:

Definition 4.6. The **higher algebraic K-groups** are defined to be the homotopy groups of $BGL(R)^+$.

$$K_n(R) := \pi_n BGL(R)^+$$

Proposition 4.7. *The definition of higher K-groups recovers the classical K-groups.*

The map $GL_n(R) \times GL_m(S) \rightarrow GL_{m+n}(R \otimes S)$ induces a map $BGL(R)^+ \wedge BGL(S)^+ \rightarrow BGL(R \otimes S)^+$ that is well-defined up to weak equivalences. This in turn induces a product on the K-groups, $K_n(R) \otimes K_m(S) \rightarrow K_{m+n}(R \otimes S)$. With the tensor product, the direct sum of all K-groups becomes a ring.

Definition 4.8. The **K-theory ring** is

$$K(R) := \bigoplus_n K_n(R)$$

with ring operations $\oplus$ and $\otimes$.

4.3. The Q Construction and Plus-Equals-Q. Using these ideas, Quillens extended the definition of K-groups to any exact category.

Definition 4.9. Let $\mathcal{C}$ be an exact category. The category QC has object $ob(\mathcal{C})$ and morphisms

$$\text{Hom}_{QC}(X, Y) := \{X \leftarrow Z \rightarrow Y\}$$

where the first arrow is an admissible epi and the second admissible mono.

Let Ω denote the loop space functor. We can apply homotopy theoretic constructions again and define the K-groups:

Definition 4.10. The K-groups of the exact category $\mathcal{C}$ are

$$K_i(\mathcal{C}) := \pi_i(\Omega BQ\mathcal{C})$$

Theorem 4.11 (+ = $\mathbb{Q}$).

$$\Omega BQ\text{Proj}(\mathbb{R}) \simeq K_0(\mathbb{R}) \times BGL(\mathbb{R})^+$$

whence

$$K_n(\mathbb{R}) \cong K_n(\text{Proj}(\mathbb{R}))$$

for all n .

4.4. Mysteries. Algebraic K-theory is largely mysterious and notoriously hard to compute — after all, it is based on the computation of homotopy groups! Nevertheless, Quillen famously computed the K-theory of finite fields, using Adams operations:

Theorem 4.12 (Quillen '72 [10]).

$$K_{2n-1}(\mathbb{F}_q) = \mathbb{Z}/(q^n - 1); K_{2n}(\mathbb{F}_q) = 0, \text{ where } n > 0.$$

The technique involved has since refused any attempt of generalization.

Assuming the *Kummer-Vandiver conjecture* (which is a statement about the class numbers of cyclotomic fields), the K-theory of the integers was computed to be $K_0(\mathbb{Z}) \cong \mathbb{Z}$, $K_1(\mathbb{Z}) \cong \mathbb{Z}/2$, and

$$K_n(\mathbb{Z}) \cong \begin{cases} 0 & \text{if } n > 0 \text{ and } n \equiv 0 \pmod{4} \\ \mathbb{Z} \oplus \mathbb{Z}/2 & \text{if } n \equiv 1 \pmod{8} \text{ and } n > 1 \\ \mathbb{Z}/c_k \oplus \mathbb{Z}/2 & \text{if } n \equiv 2 \pmod{8} \\ \mathbb{Z}/8d_k & \text{if } n \equiv 3 \pmod{8} \\ \mathbb{Z} & \text{if } n \equiv 5 \pmod{8} \\ \mathbb{Z}/c_k & \text{if } n \equiv 6 \pmod{8} \\ \mathbb{Z}/4d_k & \text{if } n \equiv 7 \pmod{8}, \end{cases}$$

where c_k/d_k is the Bernoulli number B_{2k}/k in lowest terms and $n = 4k - 1$ or $4k - 2$. We see that the 8-fold periodicity mysteriously appearing again.

Finally, the Quillen-Lichtenbaum conjecture posits that K-theory should be a very fine approximation of R using abelian groups. In more precise terms, there should be a SS beginning in the étale cohomology of $\text{Spec} R$ and converging in the K-theory of R . Vladimir Voevodsky proved this conjecture using techniques from $\mathbf{A}^1$ -homotopy, for the latter of which he was awarded the Fields medal.

Theorem 4.13 (Quillen-Lichtenbaum conjecture, proven by Voevodsky). *If A is a finitely-generated $\mathbb{Z}$ -algebra and I is prime, then there is a spectral sequence analogous to the Atiyah-Hirzebruch spectral sequence, starting at*

$$E_2^{p,q} = H_{\text{etale}}^p(\text{Spec } A[\ell^{-1}], \mathbb{Z}_\ell(-q/2))$$

and abutting to

$$K_{-p-q} A \otimes \mathbb{Z}_\ell$$

for $-p - q > 1 + \dim A$

5. WHERE DO WE GO FROM HERE?

This is a rather hand-wavy section. See [3] for all references.

We have seen that K-theory does not see through Morita equivalence, meaning it does not see the original ring at all but just its category of modules. We have also seen Quillen's Q-construction extending the definition to exact categories, so it is very tempting to come up with a "minimal" characterization of when our constructions work. A nice idea is that K-theory is the "universal machine for additive invariants." Namely, given a "three-term relation"

$$0 \rightarrow A \rightarrow C \rightarrow B \rightarrow 0,$$

K-theory sees

$$[C] = [A] + [B].$$

In other words, if we think of A and B as building blocks, K-theory tells us that [C] is the *assembly* of A and B. This allows us to consider many other settings.

Remark 5.1. We ought to use Waldhausen's $s_\bullet$ -construction for these generalizations, but the ignorant author has been too lazy to delve into it. :(

5.1. Finite Sets. When we apply the $s_\bullet$ -construction to the category of finite sets **FinSets**, we see that

$$K_0(\mathbf{FinSets}) \cong \mathbb{Z}$$

with the size of a finite set associated to each natural number, and

$$K_1(\mathbf{FinSets}) \cong \mathbb{Z}/2$$

which is precisely the abelianization of the infinite symmetric group (corresponding to the sign of any permutation). The K-theory space for finite sets is

$$\Omega|\mathbb{S}_\bullet \mathbf{FinSets}| \cong QS^0$$

which is the 0-space of the **sphere spectrum** $\mathbb{S}$. The homotopy groups of this space is exactly the **stable homotopy groups of spheres**. In other words, we recover the most fundamental object of stable homotopy theory from the K-theory construction over finite sets.

5.2. Varieties. Let $\mathbf{Var}_k$ be the category of varieties (not necessarily irreducible!) over k . We can already apply the K_0 construction on varieties. If $Y \subseteq X$ is a closed subvariety, then the three-term relation we should impose is

$$[X] = [Y] + [X \setminus Y].$$

$K_0(\mathbf{Var}_k)$ is known as the **Grothendieck ring of varieties**. The Waldhausen $s_\bullet$ -construction yields $K(\mathbf{Var}_k)$, the K-theory of varieties, where the higher K-groups capture information about groups of *birational automorphisms*.

5.3. Commutative Squares. We have been looking at three-term relations in the form of

$$A \hookrightarrow C \twoheadrightarrow B,$$

but in many categories, we also see many squares:

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

In this recent paper [8], Jonathan Campbell et.al. defined the K-construction for four-term relations

$$[A] + [D] = [B] + [C]$$

and proved that the squares K-theory for complete varieties is the same as $K(\mathbf{Var}_k)$. Lots of cool stuff!

5.4. Polytopes, Scissors Congruence, Cut-and-Paste Manifolds. Instead of assembling modules, why don't we do something much more concrete and assemble polytopes? Even simpler, what can we say about piecing together polygons in the plane? The classical **scissors congruence** problem is the following:

Given two polygons with the same area, can we always divide one into polygonal pieces and reassemble them to form the other?

If that is possible, then we say that the two polygons are *scissors congruent*.

The answer for this easiest case is YES, and similar ideas has been lurking around since the time of the Greeks. The proof was formalized by Gerhard Hessenberg in the early 1900s. Here is the idea:

- A polygon always admits a triangulation.
- Two triangles of the same area are always scissors congruent.

Hence, one first subdivide the polygons into an equal number of triangles, do the transformation on each triangle, and glue everything back together.

What if we go to higher dimensions? Well, the problem becomes difficult already in our familiar $\mathbb{R}^3$, as its name strongly suggests:

Hilbert's Third Problem: *Are two polyhedra in Euclidean 3-space of always scissors congruent if they have the same volume?*

To solve this problem, Max Dehn invented what is now known as the Dehn invariant, and his answer was NO.

Theorem 5.2 (Dehn-Sydler). *Two polyhedra in $\mathbb{R}^3$ are scissors congruent iff they have the same volume and Dehn invariant.*

For higher $\mathbb{R}^n$, the situation soon gets out of hand, and the **generalized Hilbert's Third Problem** is still an active area of research. In specific, we can make the following definition which is analogous to K_0 :

Definition 5.3. The **scissors congruence group** $\mathcal{P}(X, G)$ where G is a subgroup of the group of isometries of X , is the free abelian group on symbols $[P]$, for all polytopes in X modulo the relations (i) $[P] - [P'] - [P'']$ when $P = P' \sqcup P''$, (ii) $[gP] - [P]$.

In [9], Inna Zakharevich extended this definition to a full fledged K-theory using what are called *assemblers*. It turns out that this construction can be further generalized to *cut-and-paste manifolds*, where the correct picture to have in mind is “gluing pairs of pants” in the theory of cobordisms. This line of thought has apparently received much attention from the quantum field theory literature, about which the ignorant author knows nothing.

With the above, the author will end this leisurely excursion into the field of algebraic K-theory.

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