Intersection Theory Notes 05/13
Variations on a Theme of 27 Lines

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Contents

1 Variation 0: Introduction to 27 Lines

Throughout this write-up, let $\mathbf{k}$ be an algebraically closed field with $\operatorname{char} \mathbf{k} \neq 2$.

1.1 Cubic Surfaces

Definition 1.1. A **cubic surface** in $\mathbb{P}^3$ is a degree 3 hypersurface.

Example 1.2. The **Fermat cubic** is defined by the homogeneous equation

$$x^3 + y^3 + z^3 + w^3 = 0.$$

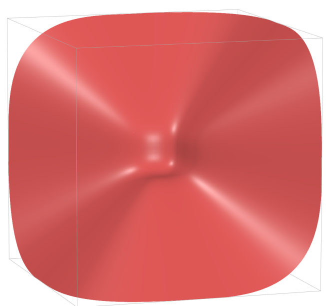


Figure 1.3: Fermat cubic with affine equation $x^2 + y^2 + z^2 + 1 = 0$, plotted in Desmos 3D.

The **Clebsch surface** is another smooth cubic surface defined by the following:

$$x^3 + y^3 + z^3 + w^3 = (x + y + z + w)^3.$$

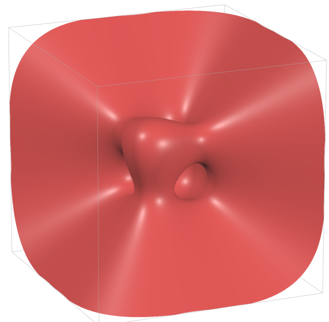


Figure 1.4: Clebsch surface.

Cayley's nodal cubic is a singular cubic surface given by the following:

$$xyz + yzw + zwx + wxy = 0.$$

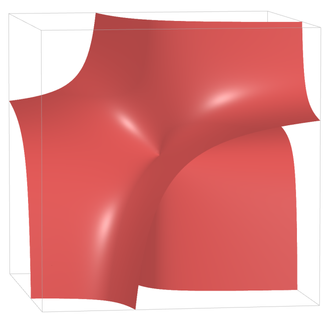


Figure 1.5: Cayley's nodal cubic.

We remember that a line in $\mathbb{P}^3$ is a codimension 2 subvariety of degree 1, i.e. a copy of $\mathbb{P}^1$.

Theorem 1.6 (Cayley 1848)**.** *Every smooth cubic surface in $\mathbb{P}^3$ contains exactly 27 distinct lines.*

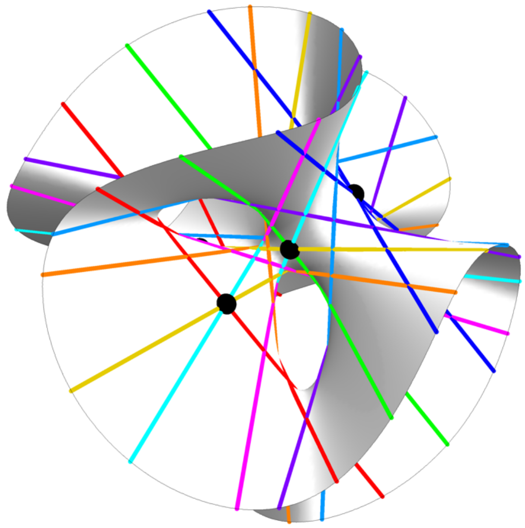


Figure 1.7: The 27 distinct lines on the Clebsch cubic, by Greg Egan. https://blogs.ams.org/visualinsight/2016/02/15/27-lines-on-a-cubic-surface/

Figure 1.8: A 3D-printed model of the Clebsch cubic with 27 lines on it.

1.2 Where are the 27 lines?

Here, we describe how to locate the 27 lines on any smooth cubic surface.

Theorem 1.9. *Every cubic surface in $\mathbb{P}^3$ is isomorphic to $\mathbb{P}^2$ blown up at 6 points.*

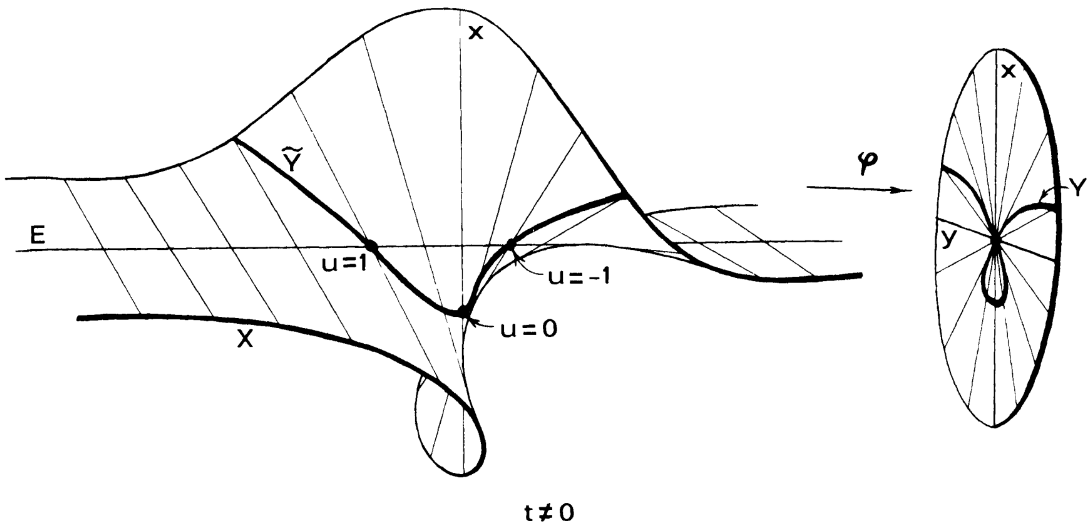


Figure 1.10: Hartshorne's famous picture of a blow-up.

This fact is reflected in the Hodge diamond of a cubic surface:

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 7 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

where the middle Hodge number $h^{1,1}$ is $7 = 1 + 6$, reflecting the fact that we erect 6 exceptional divisors over $\mathbb{P}^2$. One can in fact see all those 7 holes in the 3D model of the Clebsch cubic! Pieter Belmans (University of Luxembourg) has a nice interactive tool for computing hodge numbers: `https://pbelmans.ncag.info/`

cohomology-tables/. Pieter's webpage contains many more awesome gadgets, and we highly recommend that the reader check those out.

Now, each exceptional divisor E_i is obviously a copy of $\mathbb{P}^1$, so we have found 6 of the lines. Classical results about the intersection theory on surfaces tells us that exceptional divisors are (-1)-curves, in the sense that

$$\deg[E_i]^2 = -1.$$

The next big observation is that blowing up at more than one point creates new (-1)-curves:

Example 1.11. $\mathbb{P}^2$ blown up at 2 distinct points, say P_1 and P_2. Let E_1, E_2 be the two exceptional divisors, and let L_{12} be the line $\bar{P_1 P_2}$.

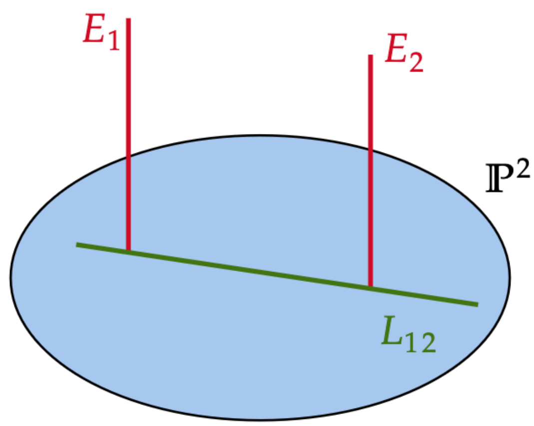


Figure 1.12: $\mathbb{P}^2$ blown up at 2 points.

Let l be the line L_{12} before blowing up. We have

$$[l]^2 = 1 \cdot [pt] \in A^2(\mathbb{P}^2).$$

After blowing up, we see that the divisor l becomes $E_1 + E_2 + L_{12}$, and as a result,

$$\begin{aligned}
1 &= \deg[l]^2 \\
&= \deg[E_1 + E_2 + L_{12}]^2 \\
&= \deg[E_1]^2 + [E_2]^2 + 2[E_1]\cdot[E_2] + 2[L_{12}]\cdot[E_1] + 2[L_{12}]\cdot[E_2] + [L_{12}]^2 \\
&= (-1) + (-1) + 2\cdot 0 + 2\cdot 1 + 2\cdot 1 + \deg[L_{12}]^2 \\
&= 2 + \deg[L_{12}]^2.
\end{aligned}$$

We see that $\deg[L_{12}]^2 = -1$! Thus, by blowing up, we have created a new (-1)-curve.

We see that by repeatedly blowing up, we gain a new line through each pair of blown-up points. Blowing up 6 times, we get another

$$\binom{6}{2} = 15$$

new lines.

Via a very similar argument, it can be shown that the unique conic passing through 5 of the blown-up points will be a (-1)-curve. Thus, we get another

$$\binom{6}{5} = 6.$$

Combining all the lines we have found together, we obtain the target count

$$6 + 15 + 6 = 27!$$

We ought to mention that the configuration of these 27 exhibit an interesting symmetry. Namely, the graph of pairwise intersections of these lines is the **Schläfli graph**. Its automorphism group is the famous exceptional reflection group E_6.

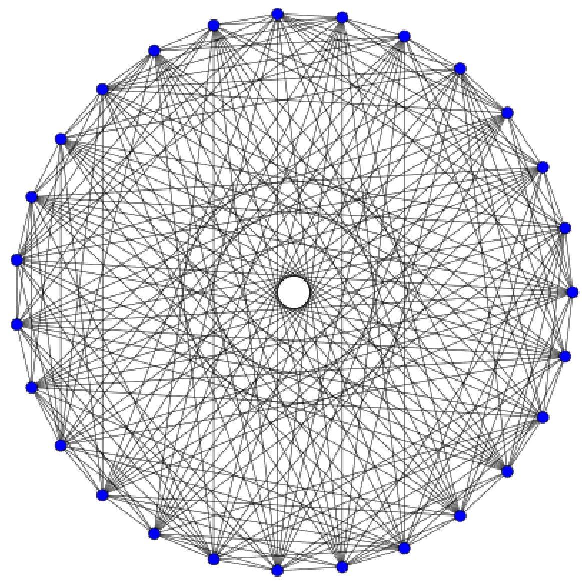


Figure 1.13: The Schläfli graph.

Of course, the real challenge comes from proving that there are always exactly these 27 distinct lines given any smooth cubic surface. One can be a bit more ambitious and ask if this count extends to singular cubic surfaces where we need to count with the correct multiplicities. We are going to do all that with the machinery of intersection theory.

2 Translating the Enumerative Problem

We will translate the enumerative problem into the degree calculation of loci in a suitable parameter space and then verify transversality.

Problem 2.1. How many lines lie on a smooth cubic surface in $\mathbb{P}^3$?

The parameter space for lines in $\mathbb{P}^3$ is $\mathbb{G}(1,3)$, within which we would like to find the locus of lines contained in a given smooth cubic surface. A smooth (in fact, any) cubic surface S is given by a homogeneous cubic polynomial F. If S contains a line l, then every point on l needs to satisfy $F = 0$, which just means $F|_l = 0$.

Now, the tautological bundle $\mathcal{S}$ over $\mathbb{G}(1,3)$ has fiber l itself over each l. Thus, the dual of the tautological bundle $\mathcal{S}^\vee$ has fiber $l^\vee$, the space of linear forms over l. Then, cubic forms over each l piece together as the symmetric cube bundle $\operatorname{Sym}^3 \mathcal{S}^\vee$ over $\mathbb{G}(1,3)$. Then, $l \mapsto F|_l$ assigns to each $l \in \mathbb{G}(1,3)$ the cubic form $F|_l$ over l, so F induces a section of the bundle $\operatorname{Sym}^3 \mathcal{S}^\vee$. Then, the locus where $F|_l = 0$ is the same as the vanishing locus of the section $l \mapsto F|_l$. Since F is taken generically, the class of the **locus of lines contained in** S is the same as the class of the **vanishing locus of a generic section** of $\operatorname{Sym}^3 \mathcal{S}^\vee$.

So we have reduced the enumerative problem to finding the vanishing locus of a generic section of $\operatorname{Sym}^3 \mathcal{S}^\vee$. We have to make sure that this locus will indeed have dimension 0, so it becomes a union of points. The stalk of $\operatorname{Sym}^3 \mathcal{S}^\vee$ at l is the space of cubic forms over $l \cong \mathbb{P}^1$, i.e.

$$(\operatorname{Sym}^3 \mathcal{S}^\vee)_l \cong \Gamma(\mathbb{P}^1, \mathcal{O}(3))$$

as $\mathbf{k}$-vector spaces, the latter of which has dimension 4 (spanned by x^3, x^2y, xy^2, y^3). Thus, $\operatorname{Sym}^3 \mathcal{S}^\vee$ is a rank 4 bundle over the base $\mathbb{G}(1,3)$ which is also of dimension 4! This confirms that the vanishing locus of a section is indeed a union of points.

Recall that for a rank-r bundle over X, the ith Chern class c_i corresponds to the degeneracy locus where $r - i$ global sections become linearly dependent. In specific, the top Chern class c_r when $r = \dim X$ corresponds to the zero locus of a generic section. Therefore, we have reduced our enumerative problem to computing the degree of the top Chern class of a bundle:

$$\text{Number of lines on a smooth cubic surface} = \deg c_4(\operatorname{Sym}^3 \mathcal{S}^\vee).$$

In the following sections, we will see a number of different ways to do this computation.

3 Variation 1: Schubert Calculus

3.1 27 Lines via Schubert Calculus

We know the Chern classes of the dual tautological bundle:

$$c(\mathcal{S}^\vee) = 1 + \sigma_1 + \sigma_{1,1}.$$

According to the splitting principle, we may pretend that $\mathcal{S}^\vee$ splits as a direct sum of line bundles, say $\mathcal{L} \oplus \mathcal{M}$, with total Chern classes $c(\mathcal{L}) = 1 + \alpha$ and $c(\mathcal{M}) = 1 + \beta$. Then,

$$c(\mathcal{S}^\vee) = c(\mathcal{L})c(\mathcal{M}) = 1 + \alpha + \beta + \alpha\beta$$

implies

$$\sigma_1 = \alpha + \beta, \sigma_{1,1} = \alpha\beta.$$

Now,

$$\operatorname{Sym}^3 \mathcal{S}^\vee = \mathcal{L}^3 \oplus (\mathcal{L}^2 \otimes \mathcal{M}) \oplus (\mathcal{L} \otimes \mathcal{M}^2) \oplus \mathcal{M}^3,$$

so

$$\begin{aligned}
c(\operatorname{Sym}^3 \mathcal{S}^\vee) &= c(\mathcal{L}^3)c(L^2 \otimes \mathcal{M})c(\mathcal{L} \otimes \mathcal{M}^2)c(\mathcal{M}^3) \\
&= (1 + 3\alpha)(1 + 2\alpha + \beta)(1 + \alpha + 2\beta)(1 + 3\beta) \\
&= (1 + 3(\alpha + \beta) + 9\alpha\beta)(1 + 3(\alpha + \beta) + 2(\alpha + \beta)^2 + \alpha\beta) \\
&= (1 + 3\sigma_1 + 9\sigma_{1,1})(1 + 3\sigma_1 + 2\sigma_1^2 + \sigma_{1,1}) \\
&= (1 + 3\sigma_1 + 9\sigma_{1,1})(1 + 3\sigma_1 + 2\sigma_2 + 3\sigma_{1,1}).
\end{aligned}$$

In particular,

$$\begin{aligned} c_4(\operatorname{Sym}^3 \mathcal{S}^\vee) &= 9\sigma_{1,1} \cdot 3\sigma_{1,1} \\ &= 27\sigma_{2,2}, \end{aligned}$$

whence

$$\deg c_4(\operatorname{Sym}^3 \mathcal{S}^\vee) = 27!$$

cf. Eisenbud and Harris 2016. Schubert calculus tells us that there are indeed 27 lines on a cubic surface when counted with correct multiplicities. How do we make sure that there cannot be double (i.e. unreduced) lines when S is smooth? For that, we need to introduce the machinery of Fano schemes.

4 Variation 2: Fano Schemes

Fano schemes are schemes parametrizing lines (and more generally, k-planes) contained on a hypersurface. They are examples of Hilbert schemes, which parametrizes flat families of subvarieties with a given Hilbert polynomial.

4.1 Definition of a Fano Scheme

To start off, we consider all degree d hypersurfaces in $\mathbb{P}^n$. Set $N = \binom{n+d}{d} - 1$, so that degree d hypersurfaces (really, homogeneous degree d polynomials in $n+1$ variables) are parametrized by $\mathbb{P}^N$. Then, the incidence correspondence of k-planes contained in a degree d hypersurface, namely

$$\Phi(n,d,k) = \left\{(X,L) \in \mathbb{P}^N \times \mathbb{G}(k,n) \mid L \subset X\right\}$$

cuts out a subvariety of $\mathbb{P}^N \times \mathbb{G}(k,n)$.

Definition 4.1. The **universal Fano scheme** of k-planes on hypersurfaces of degree d in $\mathbb{P}^n$ is the subvariety

$$\Phi(n,d,k) = \left\{(X,\Lambda) \in \mathbb{P}^N \times \mathbb{G}(k,n) \mid \Lambda \subset X\right\}.$$

Proposition 4.2. *The universal Fano scheme $\Phi = \Phi(n,d,k) \subset \mathbb{P}^N \times \mathbb{G}(k,\mathbb{P}^n)$ is a smooth irreducible variety of dimension*

$$\dim \Phi(n,d,k) = N + (k+1)(n-k) - \binom{k+d}{d}.$$

Proof

The dimension of $\mathbb{G}(k, n)$ is $(k+1)(n-k)$. The fiber of Φ at a point over $\mathbb{G}(k, n)$ is the projectivization of the kernel of the restriction map

$$\Gamma(\mathbb{P}^n, \mathcal{O}(d)) \to \Gamma(\Lambda, \mathcal{O}(d))$$

which has dimension $N - \binom{k+d}{d}$. Hence,

$$\dim \Phi(n, d, k) = (k+1)(n-k) + N - \binom{k+d}{d}.$$

■

From this, we expect the dimension of the fiber of Φ over $\mathbb{P}^N$ to be $(k+1)(n-k) - \binom{k+d}{d}$, which is not always positive!

Corollary 4.3. *(a) The dimension of any component of the family of k-planes on any hypersurface of degree d in $\mathbb{P}^n$ is at least*

$$\varphi(n, d, k) := (k+1)(n-k) - \binom{k+d}{k}.$$

(b) If $\varphi(n, d, k) < 0$, then the general hypersurface of degree d in $\mathbb{P}^n$ contains no k-planes.

(c) If $\varphi(n, d, k) \geqslant 0$ and the general hypersurface of degree d contains any k-planes, then every hypersurface of degree d contains k-planes, and every component of the family of k-planes on a general hypersurface of degree d has dimension exactly $\varphi(n, d, k)$.

We move on to define the parameter space for lines on a *given* hypersurface.

Definition 4.4. We define the **Fano scheme** $F_k(X)$ of k-planes on a hypersurface X using local coordinates. Suppose X is cut out by the homogeneous polynomial g. For a k-plane $\Lambda \xleftarrow{\alpha} \mathbb{P}^k$, choose local homogeneous coordinates $z_0, \ldots, z_k$ so that we may pull back g and expand in these coordinates:

$$\alpha^* g = \sum_{I \in \binom{[k]}{d}} c_I z^I.$$

The coefficients c_I are polynomials in local coordinates of $\mathbb{G}(k, n)$, which we may take as local equations for $F_k(X)$.

Exercise 4.5. Check that these equations agree on overlaps.

The Fano scheme is smooth and reduced for *generic* X, but for particular X it may be singular and non-reduced. (see Eisenbud & Harris pp. 197, 199)

Generalizing our translation exercise in Section 2, all Fano schemes can be described as the **vanishing loci of generic sections**, which helps us identify their classes with top Chern classes.

Proposition 4.6. *Let V be an $(n+1)$-dimensional vector space, and let $\mathcal{S} \subset V \otimes \mathcal{O}_{\mathbb{G}}$ be the tautological rank- $(k+1)$ subbundle on the Grassmannian $G = \mathbb{G}(k, \mathbb{P}V)$ of k-planes in $\mathbb{P}V \cong \mathbb{P}^n$. A form g of degree d on $\mathbb{P}V$ gives rise to a global section σ_g of $\operatorname{Sym}^d \mathcal{S}^\vee$ whose zero locus is $F_k(X)$, where X is the hypersurface $g = 0$. Thus, when $F_k(X)$ has expected codimension $\binom{k+d}{k} = \operatorname{rank}\left(\operatorname{Sym}^d \mathcal{S}^\vee\right)$ in G, we have*

$$[F_k(X)] = c_{\binom{k+d}{d}}\left(\operatorname{Sym}^d \mathcal{S}^\vee\right) \in A(G).$$

Proof

The sheaf morphism $V^\vee \otimes \mathcal{O}_{\mathbb{G}} \to \mathcal{S}^\vee$ on the stalk at $\Lambda \in \mathbb{G}(k, n)$ sends a linear form φ to its restriction $\varphi|_\Lambda$. This induces another morphism

$$\operatorname{Sym}^d V^\vee \cong \Gamma(\mathbb{P}^n, \mathcal{O}(1)) \to \operatorname{Sym}^d \mathcal{S}^\vee$$

taking a degree d form g to its restriction $g|_\Lambda$, as desired. ∎

4.2 Hilbert Schemes

Theorem 4.7 (Definition-Proposition of Hilbert Schemes)**.** *Let $X \subset \mathbb{P}^n$ be a closed subscheme, and let $P(d)$ be a polynomial. There exists a unique scheme $\mathcal{H}_P(X)$, called the **Hilbert scheme** of X for the Hilbert polynomial P, with a flat family*

$$\mathcal{H}_P(X) \times X \supset \mathcal{Y} \xrightarrow{\pi} \mathcal{H}_P(X)$$

*of subschemes of X, called the **universal family of subschemes** of X with Hilbert polynomial P, having the following properties:*

- *The fibers of π all have Hilbert polynomial equal to $P(d)$.*
- *For any flat family*

  $$B \times X \supset \mathcal{Y}' \xrightarrow{\pi} B$$

  whose fibers have Hilbert polynomial $P(d)$, there is a unique morphism $\alpha : B \to \mathcal{H}_P(X)$ such that $\mathcal{Y}'$ is equal to the pullback of $\mathcal{Y}$:

$$\begin{tikzcd}
\mathcal{Y}' \arrow[r, "="] \arrow[dr] & B \times_{\mathcal{H}_P} \mathcal{Y} \arrow[r, dashed] \arrow[d, dashed] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{Y} \arrow[d, "\pi"] \\
& B \arrow[r, dashed, "!\alpha"] & \mathcal{H}_P
\end{tikzcd}$$

The Hilbert scheme $\mathcal{H}_P(X)$ represents the following functor from $\mathbf{Sch_k}$ to **Set**:

$$F_{X,P} : B \mapsto \{ \text{flat families } X \times B \supset \mathcal{Y}' \to B \text{ of subschemes of } X \subset \mathbb{P}^n$$
$$\text{whose fibers over closed points all have Hilbert polynomial } P\}$$

Proposition 4.8. *There is a natural isomorphism*

$$F_{X,P} \cong Hom_{\mathbf{Sch_k}}(-, \mathcal{H}_P).$$

In these terms, Grassmannians are Hilbert schemes of k-planes in $\mathbb{P}V$, while Fano schemes are Hilbert schemes of k-planes on X, each with Hilbert polynomial $\binom{d+k}{k}$.

4.3 Tangent Spaces as First Order Deformations

Recall, we need to verify that the 27 lines are distinct for a smooth cubic surface S. Since lines on S are points of $F_1(S)$, this is to say that $F_1(S)$ contains no double points. Thus, we need to check that the scheme $F_1(S)$ is *reduced.*

Since $F_1(S)$ is zero-dimensional, reducedness is the same as smoothness. In order to show that $F_1(S)$ is smooth for a smooth S, we need to study its tangent spaces. Thankfully, tangent spaces over a Hilbert scheme have particularly nice descriptions using the idea of deformations, vastly generalizing the identification $\mathcal{T}_G \cong \mathscr{H}om(\mathcal{S}, \mathcal{Q})$ in the Grassmannians chapter.

Definition 4.9. Let $Y \subseteq X$ be a subscheme. Let T be a 'parametrizing' subscheme with a distinguished $\mathbf{k}$-point $\operatorname{Spec} \mathbf{k} \to T$. A **deformation of** Y **in** X **over** T **with distinguished point** $\operatorname{Spec} \mathbf{k}$ is a subscheme $\mathcal{Y} \subseteq T \times X$ which is flat over T, whose fiber over the distinguished point is Y.

$$\begin{tikzcd}
Y \arrow[r] \arrow[d] & \mathcal{Y} \arrow[r, hook] \arrow[d, "\text{flat}"'] & T \times X \arrow[dl, "\pi_T"] \\
\operatorname{Spec} \mathbf{k} \arrow[r] & T &
\end{tikzcd}$$

A deformation is called a **first-order deformation** if T looks the spectrum of a local ring,

$$T_m := \operatorname{Spec} R_m = \operatorname{Spec} \mathbf{k}\left[\epsilon_1, \ldots, \epsilon_m\right] / \left(\epsilon_1, \ldots, \epsilon_m\right)^2.$$

The scheme T_m is often illustrated as a 'fuzzy' point with m 'infinitesimal vectors.' We think of T_m as a model for the Zariski tangent space at a smooth point on an m-dimensional scheme. This idea has shown up in the 'sheafy proof' for the tangent space of a Grassmannian, in the previous set of notes.

The following proposition holds by the universal property of Hilbert schemes.

Proposition 4.10. *Let H be the Hilbert scheme of Y in X. Let $0 \in T_m$ denote the unique closed point. Then, we have the set bijection*

$$\{\text{deformations of } Y \subset X \text{ over } T_m\} \quad \stackrel{\sim}{\longleftrightarrow} \quad \{\varphi : T_m \to H \mid \varphi(0) \mapsto Y\}.$$

We can make another identification of first-order deformations:

Proposition 4.11. *There is a set bijection between first order deformations and $\mathcal{O}_Y$-module morphisms*

$$\{\text{deformations of } Y \subset X \text{ over } T_m\} \quad \stackrel{\sim}{\longleftrightarrow} \quad \operatorname{Hom}_{\mathcal{O}_Y}(\mathcal{I}_Y/\mathcal{I}_Y^2, \mathcal{O}_Y^{\oplus m}).$$

Proof

Reduce to the case where X, Y are affine and use the characterization of flatness over the local ring R_m. (See Eisenbud & Harris pp. 214-215) ■

For the next implication, set $m = 1$ so as to recover the alternate definition of tangent spaces. Then, Proposition 4.10 specializes as

$$\{\text{deformations of } \; Y \subset X \text{ over } T_1\} \quad \xleftrightarrow{\sim} \quad \{\varphi : T_1 \to H \mid \varphi(0) \mapsto Y\} \cong \mathcal{T}_{[Y]}H.$$

Composing the previous two identifications, we conclude the following characterization of tangent spaces to a Hilbert scheme:

Theorem 4.12 (Tangent Space of Hilbert Schemes)**.** *Suppose that $Y \subset X$ is a subscheme of a -scheme $X \subset \mathbb{P}^n$, and let H be the Hilbert scheme of Y. If $[Y] \in H$ denotes the point corresponding to Y, then*

$$T_{[Y]/H} \cong \Gamma\left(\mathscr{H}om_{\mathcal{O}_Y}\left(\mathcal{I}_{Y/X}/\mathcal{I}^2_{Y/X}, \mathcal{O}_Y\right)\right)$$

as vector spaces.

One realizes that the $\mathscr{H}om$ term in this formula is in fact the normal bundle of Y in X:

Proposition 4.13 (Proposition-Definition of Normal Bundles)**.** *Suppose that $Y \subset X$ are schemes.*

(a) If X and Y are smooth varieties then $\mathcal{N}_{Y/X} = \mathscr{H}om_{\mathcal{O}_Y}\left(\mathcal{I}_Y/\mathcal{I}_Y^2, \mathcal{O}_Y\right)$. For arbitrary schemes $Y \subset X$, we define $\mathcal{N}_{Y/X}$ by this formula.

(b) If $Y \subset X \subset W$ are schemes, and X is locally a complete intersection in W, then there is a left exact sequence of normal bundles

$$0 \longrightarrow \mathcal{N}_{Y/X} \longrightarrow \mathcal{N}_{Y/W} \xrightarrow{\alpha} \mathcal{N}_{X/W}\Big|_Y .$$

If all three schemes are smooth, then α is an epimorphism.

(c) If Y is a Cartier divisor on X then $\mathcal{N}_{Y/X} = \mathcal{O}_X(Y)$. More generally, if Y is the zero locus of a section of a bundle $\mathcal{E}$ of rank e on X, and Y has codimension e in X, then

$$\mathcal{N}_{Y/X} = \mathcal{E}|_Y$$

For part (c), if Y is a complete intersection of X with divisors on $\mathbb{P}^n$ of degrees d_i, then we recover our familiar result: $\mathcal{N}_{Y/X} = \bigoplus \mathcal{O}_X(d_i)$.

Proof

(a)For any inclusion of subschemes $Y \subset X$, there is a right exact sequence involving the cotangent sheaves of X and Y :

$$\mathcal{I}_{Y/X}/\,\mathcal{I}_{Y/X}^2 \xrightarrow{d} \Omega_X\Big|_Y \longrightarrow \Omega_Y \longrightarrow 0,$$

where d is the map taking the class of a (locally defined) function $f \in \mathcal{I}_{Y/X}$

to its differential $df \in \Omega_X|_Y$. If X and Y are smooth, then Y is locally a complete intersection in X, so $\mathcal{I}_{Y/X}/\mathcal{I}^2_{Y/X}$ is a locally free sheaf on Y of rank equal to $\dim X - \dim Y = \operatorname{rank} \Omega_X|_Y - \operatorname{rank} \Omega_Y$, i.e. it is of full rank. Thus d is a monomorphism, and the RES turns into a SES

$$0 \to \mathcal{I}_{Y/X}/\,\mathcal{I}^2_{Y/X} \xrightarrow{d} \Omega_X\Big|_Y \to \Omega_Y \to 0.$$

Since Y is smooth, Ω_Y is locally free, so dualizing preserves exactness, and we get an exact sequence

$$0 \longleftarrow \mathscr{H}om_{\mathcal{O}_Y}\left(\mathcal{I}_{Y/X}/\mathcal{I}^2_{Y/X}, \mathcal{O}_Y\right) \longleftarrow \mathcal{T}_X\Big|_Y \longleftarrow \mathcal{T}_Y \longleftarrow 0,$$

where the right-hand map is the differential of the inclusion $Y \subset X$. Comparing with the normal bundle sequence

$$0 \to \mathcal{T}_Y \to \mathcal{T}_X|_Y \to \mathcal{N}_{Y/X} \to 0,$$

we conclude that $\mathcal{N}_{Y/X} = \mathscr{H}om_{\mathcal{O}_Y}\left(\mathcal{I}_Y/\mathcal{I}^2_Y, \mathcal{O}_Y\right)$. Proofs to the other parts are left as an exercise. ■

Therefore, we can identify the Hom term as the normal bundle and specialize to Fano schemes. As a corollary to Theorem 4.12, we deduce that the tangent space to $F_k(X)$ at $[L]$ is isomophic to global sections of the normal bundle of L in X, i.e. normal vector fields along L in X:

Theorem 4.14 (Tangent Space of Fano Schemes)**.** *Suppose that $L \subset X$ is a k-plane in a smooth variety $X \subset \mathbb{P}^n$, and let $[L] \in F_k(X)$ be the corresponding point. The Zariski tangent space of $F_k(X)$ at $[L]$ is $\Gamma(\mathcal{N}_{L/X})$.*

The result is intuitively plausible if we think of a section of $\mathcal{N}_{L/X}$ as providing a normal vector at each point in X, with a corresponding infinitesimal motion of X. Eisenbud & Harris has a nice illustration of this idea:

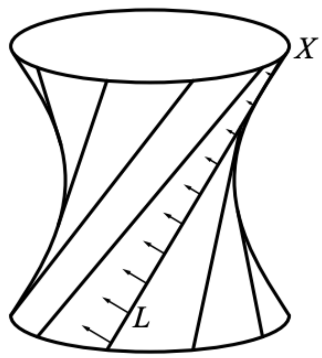


Figure 4.15: An infinitesimal movement of L in X corresponds to a normal vector field on L.

This also gives us a good criterion for the smoothness of a Fano scheme at a point.

Corollary 4.16 (Smoothness Test for Fano Schemes)**.** *Suppose that $L \subset X$ is a k-plane in a smooth variety $X \subset \mathbb{P}^n$, and let $[L] \in F_k(X)$ be the corresponding point. The dimension of $F_k(X)$ at $[L]$ is at most $\dim \Gamma\left(\mathcal{N}_{L/X}\right)$. Moreover, $F_k(X)$ is smooth at $[L]$ if and only if equality holds.*

Proof

It suffices to check locally. The dimension of the Zariski tangent space is always greater than or equal to the Krull dimension of the local ring, with equality if and only if the local ring is regular.

Smooth always implies regular, whereas regular implies smooth if the scheme is of finite type over a perfect field. $F_k(X)$ is of finite type over $\mathbf{k} = \bar{\mathbf{k}}$, which is perfect, so regular implies smooth. ■

We are now ready to check that for a smooth cubic surface S, $F_1(S)$ is reduced.

Proposition 4.17. *Let $S \subset \mathbb{P}^3$ be a smooth surface of degree $d \geqslant 3$. If $F_1(S) \neq \varnothing$, then $F_1(S)$ is smooth and zero-dimensional, thus everywhere reduced. In particular, S contains at most finitely many lines, and if $d = 3$ then S contains exactly 27 distinct lines.*

Proof

Fix $L \subseteq S$. We have seen that if degree $d \geqslant 3$, then S has negative self-intersection, so $\mathcal{N}_{L/S}$ is a line bundle with negative degree. Hence, $\dim \Gamma(\mathcal{N}_{L/S}) = 0$, which agrees with $\dim F_1(S) = 0$. By Corollary 4.16, $F_1(S)$ is reduced, whence we see that all 27 lines suggested by Schubert calculus are distinct. ■

Needless to say, our reasoning generalizes to higher dimensions:

Proposition 4.18. *The Fano scheme of lines on any smooth hypersurface of degree $d \leqslant 3$ is smooth and of dimension $2n - 3 - d$. But if $n \geqslant 4$ and $d \geqslant 4$, then there exist smooth hypersurfaces of degree d in $\mathbb{P}^n$ whose Fano schemes are singular or of dimension $> 2n - 3 - d$.*

For example, we can count the number of lines on a degree $d = 2n-3$ hypersurface in $\mathbb{P}^n$. In that case, the number is equal to

$$\deg F_1(S) = \deg c_{d+1} \operatorname{Sym}^d \mathcal{S}^\vee$$

of the symmetric power bundle over $\mathbb{G}(1,n)$. Luckily, this is built into MaCaulay2: we recommend that the reader try this on their own using this online MaCaulay2 interface! https://www.unimelb-macaulay2.cloud.edu.au/#home

```
Macaulay2, version 1.23.0.1
with packages: ConwayPolynomials, Elimination, IntegralClosure, InverseSystems, Isomorphism, LLLBases, MinimalPrimes, OnlineLoc

i1 : loadPackage "Schubert2"

o1 = Schubert2

o1 : Package

i2 : grassmannian = (m,n)-> flagBundle({m+1, n-m})

o2 = grassmannian

o2 : FunctionClosure

i3 : for n from 3 to 10 do( G=grassmannian(1,n); (S,Q) = G.Bundles; d = 2*n-3; print integral chern symmetricPower(d, dual S))
27
2 875
698 005
305 093 061
210 480 374 951
210 776 836 330 775
289 139 638 632 755 625
520 764 738 758 073 845 321
```

Figure 4.19: Web interface of Macaulay2, with the Schubert package.

The second row in the output suggests that there are 2875 lines on a smooth quintic threefold in $\mathbb{P}^4$.

4.4 Geometry of Universal Fano Scheme

We can be more ambitious and try to understand the class of the universal Fano scheme $\Phi(n,d,1)$ in $\mathbb{P}^N \times \mathbb{G}(1,n)$. This allows us to answer the following generalization of the 27 lines problem:

Problem 4.20. Let $\{X_t \subset \mathbb{P}^3\}_{t\in\mathbb{P}^1}$ be a general pencil of cubic surfaces, and consider the locus $C \subset \mathbb{G}(1,3)$ of all lines $L \subset \mathbb{P}^3$ that are contained in some member of this family. What is the genus of C ? What is the degree of the surface $S \subset \mathbb{P}^3$ swept out by these lines?

Theorem 4.21. *The universal Fano scheme $\Phi(n,d,1)|_M$ of lines on a general m dimensional linear family $M = \mathbb{P}^m$ of hypersurfaces of degree d in $\mathbb{P}^n$ is reduced and of codimension $d+1$ in the $(2n-2+m)$-dimensional space $\mathbb{P}^m \times \mathbb{G}(1,n)$. It is the zero locus of a section of the rank $-(d+1)$ vector bundle $\mathcal{E} = \pi_2^* \operatorname{Sym}^d \mathcal{S}^* \otimes \pi_1^* \mathcal{O}_{\mathbb{P}^m}(1)$ on that space, so its class is $c_{d+1}(\mathcal{E})$.*

Corollary 4.22. *The class of the universal Fano scheme $\Phi(3,3,1)$ of lines on cubic surfaces in $\mathbb{P}^3$ is*

$$
\begin{aligned}
[\Phi(3,3,1)] &= c_4\left(\pi_2^* \operatorname{Sym}^3 \mathcal{S}^* \otimes \pi_1^* \mathcal{O}_{\mathbb{P}^{19}}(1)\right) \\
&= 27\sigma_{2,2} + 42\sigma_{2,1}\zeta + (11\sigma_2 + 21\sigma_{1,1})\,\zeta^2 + 6\sigma_1\zeta^3 + \zeta^4
\end{aligned}
$$

If we take a single smooth cubic, then we are restricting this class to a point in $\mathbb{P}^{19}$, whence we recover the count $\deg[\Phi(3,3,1)|_{[S]}] = \deg 27\sigma_{2,2} \cdot \zeta^{19} = 27$ again.

Corollary 4.23. *If C is the curve of lines on a general pencil of cubic surfaces, then the degree of C is 42 and the genus of C is 70 .*

Proof

$$
\begin{aligned}
\deg C &= \deg[\Phi] \cdot \sigma_1 \cdot \zeta^1 8 \\
&= 42.
\end{aligned}
$$

∎

See Eisenbud and Harris 2016 for a much more thorough treatment.

5 Variation 3: Equivariant Integral

In section 2, we saw that the number of lines on a smooth cubic surface can be translated into the degree calculation

$$\deg c_4(\operatorname{Sym}^3 \mathcal{S}^\vee) = \int_{\mathbb{G}(1,3)} c_4(\operatorname{Sym}^3 \mathcal{S}^\vee)$$

where we may identify the degree calculation to the integral over the fundamental class via the Chow ring – de Rham cohomology isomorphism.

Problem 5.1. If Schubert calculus were never well-understood, then is there an alternate way to compute this integral?

We observe that the parameter spaces (projective spaces, Grassmannians, etc.) are highly symmetric —– namely, they all have a dense torus action. Therefore, it would be nice if we could exploit that symmetry and throw away redundant information. Thankfully, equivariant integration and its localization package comes to rescue.

5.1 Defining an Equivariant Cohomology Theory

Say we are given a space X with a group action $G \circlearrowright X$, and we would like to integrate certain cohomology classes over all of X. If we had a theory of integration that captured this group action, then it is plausible that the amount of work would significantly reduce.

Well, we have a very well understood theory of ordinary cohomology, so it is tempting to just apply that to some space related to $G \circlearrowright X$. A naive attempt is to look at the space of orbits,

$$X/G = \{xG \mid x \in X\}, \mathcal{T} = \text{quetient topology}$$

If the G-action is proper and discontinuous, then X/G has the structure of an **orbifold**.

The problem is, these spaces of orbits often contain singularities and therefore fail to be a manifold, whence singular/de Rham cohomology likely do not apply. However, that is not to say that these spaces are weird and pathological — in fact, they appear ubiquitously has singular varieties in algebraic geometry.

Example 5.2. (**Kummer surfaces**) Kummer surfaces are a family of singular surfaces which are quotients of $\mathbb{T}^4$ by $\mathbb{Z}_{/2}$. The **Roman/Steiner surface** is a member of this family, given by the projective equation

$$(x^2 + y^2 + z^2 - w^2)^2 = ((z - w)^2 - 2x^2)((z + w)^2 - 2y^2)$$

Homework: Blowing up at the origin, check that the center point is a ordinary triple point.

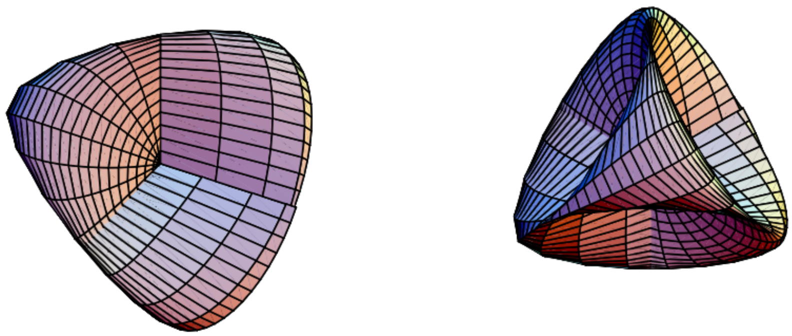


Figure 5.3: Steiner surface in 3D.

Even in cases where X/G is a genuine manifold, $H^{\bullet}(X/G)$ often remains wildly unsatisfactory. Illustrated by the following example, X/G tends to forget both X and G, let alone remembering $G \circlearrowright X$.

Example 5.4. Let $X = \mathbb{S}^2$ and $G = \mathbb{S}^1 \circlearrowright \mathbb{S}^2$ by rotation along the z-axis.

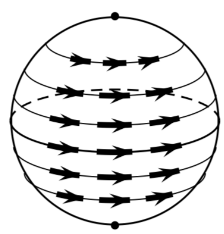


Figure 5.5: $\mathbb{S}^1$ acts on $\mathbb{S}^2$ by rotation along an axis.

The quotient $\mathbb{S}^2/\mathbb{S}^1$ is just the unit interval I, whose cohomology $H^{\bullet}(I)$ is just that of a point, since I is contractible. Therefore, we lose all information about both G and X.

The issue is the usual / **geometric quotient** is not even 'as big as' X itself, so it is silly to ask it to remember the extra information of G. Instead, we would like to find a 'bigger' space with enough room to store the G-action. It turns out that such a space, called the **homotopy quotient** $X//G$, can be constructed using the **classifying space** of G.

5.2 Classifying Spaces

The classifying space construction originates in the classification of fiber bundles with a fixed structure group G. It is a fact that every fiber bundle is a 'fiber product' of the fiber with a principal G–bundle, and the classifying space BG classifies principal G–bundles. A side product of this construction is a contractible total space EG with a free G action, which is the essential ingredient for the homotopy quotient construction.

Definition 5.6. A principal G-bundle $P \to B$ is *a* **universal bundle** if any principal G-bundle is the pullback of $P \to B$ along some continuous map, and isomorphic bundles are pullbacks along homotopic maps. The base space B is called *a* **classifying space** of G.

Proposition 5.7. *If $P \to B$ and $P' \to B'$ are universal principal G-bundles, then $B \simeq B'$.*

Proposition 5.8. *(Detection of universal bundles) If $P \to B$ is a principal G-bundle and P is contractible, then $P \to B$ is a universal bundle.*

Since all classifying spaces are homotopy equivalent, we may refer to any one of these as *the* classifying space and denote *the* universal bundle as

$$\begin{array}{c} EG \\ \downarrow \\ BG \end{array}$$

In particular, $BG \cong EG/G$, the quotient of EG by the global right action of the principal G-bundle. Then, every principal G-bundle $P \to X$ is the pullback (up to homotopy) of the **classifying map** $X \to BG$:

$$\begin{tikzcd} P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & EG \arrow[d] \\ X \arrow[r] & BG \end{tikzcd}$$

Furthermore, the following propositions relates the classifying space of a Lie group G to its closed subgroups.

Proposition 5.9. *Suppose G is a Lie group, and $H \leqslant G$ is a closed subgroup such that $G/H \simeq *$. Then, $BG \simeq BH$.*

Proposition 5.10. *Suppose G is a Lie group, and $H \leqslant G$ is a closed Lie subgroup. Then, if $EG \to BG \cong EG/G$ is the universal G-bundle, then $EG \to EG/H$ is the universal H-bundle.*

Therefore, to find the classifying space for each Lie group G, it suffices to find a principal G-bundle with contractible total space, and that will give us the classifying spaces for all of its closed Lie subgroups.

As a quintessential example, let's look at $GL(n, \mathbb{R})$, the structure group for rank-n real vector bundles.

Example 5.11. Let $G = GL(n, \mathbb{R})$. $GL(n, \mathbb{R}) \circlearrowright \Sigma_n$, the space of $n \times n$ real symmetric matrices by conjugation, with isotropy group $O(n)$. Therefore,

$$GL(n, \mathbb{R})/O(n) \cong \Sigma_n$$

but $\Sigma_n \cong \mathbb{R}^{\frac{n(n+1)}{2}}$ since each real symmetric matrix is uniquely determined by its upper-triangular entries. Therefore, it is contractible:

$$\Sigma_n \cong \mathbb{R}^{\frac{n(n+1)}{2}} \simeq *$$

Hence, the classifying spaces for $GL(n, \mathbb{R})$ and $O(n)$ are the same!

To find a principal $O(n)$-bundle with contractible total space, consider the following spaces:
The **(real) Stiefel manifold** $V_n(\mathbb{R}^k) = \{\text{orthogonal n-frames of } \mathbb{R}^k\}$, topologized as a subspace of $(\mathbb{S}^{k-1})^n$.
The **(real) Grassmannian** $Gr_n(\mathbb{R}^k) = \{n\text{-dimensional linear subspaces of } \mathbb{R}^k\}$, with the quotient topology.

$V_n(\mathbb{R}^k) \to Gr_n(\mathbb{R}^k)$ is a principal $O(n)$-bundle, where the projection map is given by taking the span of orthogonal n-frames. Then, it is a fact that every rank-n vector bundle $P \to X$ with base space of dimension k is a pullback of this bundle:

$$\begin{tikzcd} P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & V_n(\mathbb{R}^k) \arrow[d] \\ X \arrow[r] & Gr_n(\mathbb{R}^k) \end{tikzcd}$$

Since we would like to classify rank-n vector bundles over ALL base spaces, it is a common trick to consider the infinite union of these spaces:
The inclusion

$$\ldots \hookrightarrow \mathbb{R}^k \hookrightarrow \mathbb{R}^{k+1} \hookrightarrow \mathbb{R}^{k+2} \hookrightarrow \ldots$$

induces

$$\dots \hookrightarrow V_n(\mathbb{R}^k) \hookrightarrow V_n(\mathbb{R}^{k+1}) \hookrightarrow V_n(\mathbb{R}^{k+2}) \hookrightarrow \dots$$

and

$$\dots \hookrightarrow Gr_n(\mathbb{R}^k) \hookrightarrow Gr_n(\mathbb{R}^{k+1}) \hookrightarrow Gr_n(\mathbb{R}^{k+2}) \hookrightarrow \dots$$

Taking the colimit along these inclusions, we can define

$$\mathbb{R}^\infty := \varinjlim_k \mathbb{R}^k$$

$$V_n(\mathbb{R}^\infty) := \varinjlim_k V_n(\mathbb{R}^k)$$

and

$$Gr_n(\mathbb{R}^\infty) := \varinjlim_k V_n(\mathbb{R}^k)$$

endowed with the topology of an infinite union. Then, $V_n(\mathbb{R}^\infty) \to Gr_n(\mathbb{R}^\infty)$ remains a principal $O(n)$-bundle, and the shift map on $\mathbb{R}^\infty$,

$$s : \mathbb{R}^\infty \to \mathbb{R}^\infty, (x_0, x_1, x_2, \dots) \mapsto (0, x_0, x_1, \dots)$$

induces a contraction $V_n(\mathbb{R}^\infty) \simeq *$. Therefore, by 5.8, $V_n(\mathbb{R}^\infty) \to Gr_n(\mathbb{R}^\infty)$ is a universal $O(n)$-bundle, making $Gr_n(\mathbb{R}^\infty)$ the classifying space of $O(n)$. Every principal $O(n)$-bundle $P \to X$ is then the pullback of $X \to BO(n) \cong Gr_n(\mathbb{R}^\infty)$:

$$\begin{tikzcd} P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & V_n(\mathbb{R}^\infty) \arrow[d] \\ X \arrow[r] & Gr_n(\mathbb{R}^\infty) \end{tikzcd}$$

Example 5.12. A similar argument shows that $BU(n) \cong Gr_n(\mathbb{C}^\infty)$, the infinite complex Grassmannian.

We know that $GL(n, \mathbb{R})$ deformation retracts onto $O(n)$, and $GL(n, \mathbb{C})$ deformation retracts onto $U(n)$ via Gramm-Schmidt. By 5.9,

$$BGL(n, \mathbb{R}) \cong BO(n) \cong Gr_n(\mathbb{R}^\infty)$$

and

$$BGL(n, \mathbb{C}) \cong BU(n) \cong Gr_n(\mathbb{C}^\infty).$$

Example 5.13. Let $GL^+(n, \mathbb{R})$ be the subgroup of $GL(n, \mathbb{R})$ with positive determinant. Then, $GL^+(n, \mathbb{R})$ deformation retracts onto $SL(n, \mathbb{R})$ via straight-line homotopy and also onto $SO(n)$ via Gramm-Schmidt. By 5.10, we may take the same total space $EO(n, \mathbb{R}) \cong V_n(\mathbb{R}^\infty)$ and quotient out the action by the subgroup. We get $V_n(\mathbb{R}^\infty)/SO(n) \cong Gr_n^+(\mathbb{R}^\infty)$, the **oriented Grassmannian** whose points are *oriented* dim-n subspaces of $\mathbb{R}^\infty$. By 5.9, we get

$$BGL^+(n, \mathbb{R}) \cong BSL(n, \mathbb{R}) \cong BSO(n) \cong Gr_n^+(\mathbb{R}^\infty).$$

Example 5.14. In particular, consider $GL(1, \mathbb{R}) \cong \mathbb{R}^\times$ and $O(1) \cong \{\pm 1\} \cong \mathbb{Z}/2$. The rank-1 real Stiefel $V_1(\mathbb{R}^\infty)$ is just norm-1 vectors in $\mathbb{R}^\infty$, so it is $\mathbb{S}^\infty := \varinjlim_k \mathbb{S}^k$. The rank-1 real Grassmannian is just lines in $\mathbb{R}^\infty$, so it is $\mathbb{RP}^\infty := \varinjlim_k \mathbb{RP}^k$. Therefore,

$$B\mathbb{R}^\times \cong B\mathbb{Z}/2 \cong \mathbb{RP}^\infty.$$

Similarly, $GL(1, \mathbb{C}) \cong \mathbb{C}^\times$ and $U(1) \cong S^1$, the circle group. We have

$$B\mathbb{C}^\times \cong BS^1 \cong \mathbb{CP}^\infty.$$

Exercise 5.15. Q: What are the classifying spaces for the following Lie groups?

$$SL(n, \mathbb{C}), SU(n), Sp(n), Spin(n)$$

A: ____

5.3 The Borel Construction and Borel Equivariant Cohomology

cf. Anderson and Fulton 2023 and Tu 2020. It is a fact that the Cartesian product of a G–space with another that has a *free* G-action maintains the freeness of the action.

Lemma 5.16. *If a group G acts on a space E freely, then no matter how G acts on a space M, the diagonal action of G on $E \times M$, $g \cdot (e, x) = (g \cdot e, g \cdot x)$, is free.*

Furthermore, quotienting by a *free and proper* action behaves nicely with respect to taking cohomology:

Example 5.17. $G = \mathbb{Z}$ acts freely and properly on the real line $M = \mathbb{R}$ by translation: for $n \in \mathbb{Z}$ and $x \in \mathbb{R}$,

$$n \cdot x = x + n.$$

The orbit space M/G is $\mathbb{R}/\mathbb{Z} = S^1$. Its integer cohomology $H^*(M/G; \mathbb{Z}) = H^*\left(S^1\right) \cong \mathbb{Z}[x]/x^2$ contains useful information, as opposed to 5.4.

This motivates us to consider replacing the ordinary quotient by the quotient of a 'larger' space, since G acts on EG freely and properly.

Definition 5.18. For G-spaces E and M, write

$$E \times_G M := E \times M/\sim$$

where $\sim$ mods out the diagonal action: $(e, x) \sim (g \cdot e, g \cdot x)$ for all $g \in G$.

Definition 5.19. Let $e \cdot g$ denote the global right action on EG. For a G-space M, the **homotopy quotient** $M//G$ or equivalently the **Borel construction** is defined to be the quotient of $EG \times M$ by the diagonal action $g \cdot (e, x) = (e \cdot g^{-1}, g \cdot x)$:

$$M//G := EG \times_G M$$

One may also recognize the homotopy quotient as the **associated fiber bundle** to the principal bundle $EG \to BG$ with action $G \circlearrowleft M$ on the model fiber. As so, it fits into **Cartan's mixing diagram** for associated bundles:

$$\begin{tikzcd} EG \arrow[d, "\pi"] & EG \times M \arrow[l] \arrow[r] \arrow[d] & M \arrow[d] \\ BG & EG \times_G M \arrow[l, "\tau"'] \arrow[r] & M/G \end{tikzcd}$$

Cartan's mixing diagram is commutative, and π being a principal bundle implies that $\tau : EG \times_G M \to BG$ is a fiber bundle.

Example 5.20. Let's consider the action $\mathbb{S}^1 \circlearrowleft \mathbb{S}^2$ in 5.4. We recall that $B\mathbb{S}^1 = \mathbb{CP}^\infty$ with total space $E\mathbb{S}^1 = V_1(\mathbb{C}^\infty) \cong \mathbb{S}^\infty$, so the homotopy quotient is

$$\mathbb{S}^2//\mathbb{S}^1 = \mathbb{S}^\infty \times_{\mathbb{S}^1} \mathbb{S}^2$$

and it is a fiber bundle over $\mathbb{CP}^\infty$.

Definition 5.21. The **(Borel) equivariant cohomology** of X by G is defined to be the singular cohomology of the homotopy quotient $X//G$:

$$H^\bullet_G(X;R) := H^\bullet\left(X//G;R\right),$$

Remark 5.22. If the G action on X is free, then Borel cohomology indeed agrees with the singular cohomology of the space of orbits! (Hint: use Cartan's mixing diagram.)

Since $M//G$ is an associated bundle, we need a tool to compute the singular cohomology of a fiber bundle, given the cohomology of its base and fiber.

Theorem 5.23 (**Serre Spectral Sequence**)**.** *Given a homotopy fiber sequence $F \to E \to B$ over a connected topological space B, such that the canonical group action of the fundamental group $\pi_1(B)$ on the ordinary cohomology of the fiber F is trivial (for instance, if B is simply connected), then there exists a cohomology spectral sequence of the form:*

$$E_2^{p,q} = H^p\left(B, H^q(F)\right) \Rightarrow H^{p+q}(E).$$

Example 5.24. Let's return to the example $\mathbb{S}^1 \circlearrowright \mathbb{S}^2$ and calculate its integer cohomology. We have determined that the homotopy quotient fits into the following bundle

$$\mathbb{S}^2 \to \mathbb{S}^\infty \times_{\mathbb{S}^1} \mathbb{S}^2 \to \mathbb{CP}^\infty$$

We understand the cohomology of the base and the fiber:

$$H^\bullet(\mathbb{S}^2;\mathbb{Z}) \cong \mathbb{Z}[x]/x^2, |x| = 2$$

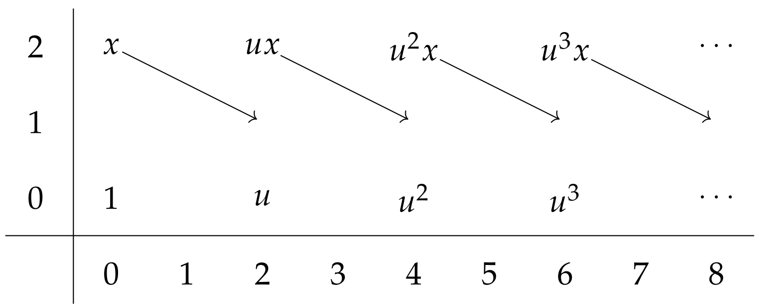
and

$$H^\bullet(\mathbb{CP}^\infty;\mathbb{Z}) \cong \mathbb{Z}[u], |u| = 2$$

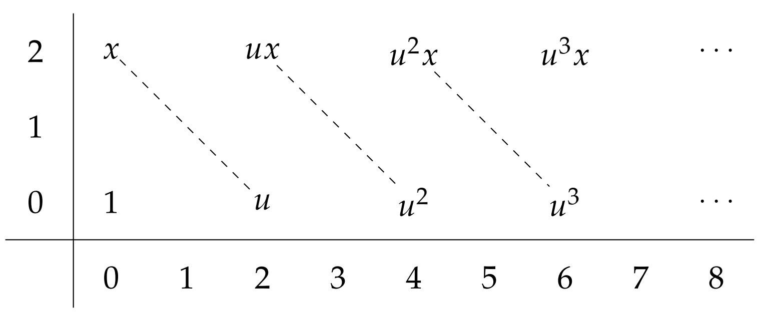
so the E_2-page of the Serre spectral sequence is

$$E^2 \cong H^\bullet(\mathbb{S}^2;\mathbb{Z}) \otimes_{\mathbb{Z}} H^\bullet(\mathbb{CP}^\infty;\mathbb{Z}) \cong \mathbb{Z}[x,u]/(x^2), |x| = |u| = 2.$$

The E_2-page looks like



We see that because the differential d_r has direction $(r, -r+1)$ and all gaps on E_2 are even, by a parity argument, no differential ever hits a none-zero class, so the spectral sequence collaspes on the E_2-page. Furthermore, since all of these modules are free, there are no none-trivial extension problems on the E_∞-page:



Thus, we have

$$H^2_{S^1}(S^2) \cong \mathbb{Z} \cdot u \oplus \mathbb{Z} \cdot x$$
$$H^4_{S^1}(S^2) \cong \mathbb{Z} \cdot u^2 \oplus \mathbb{Z} \cdot ux$$
$$H^2_{S^1}(S^2) \cong \mathbb{Z} \cdot u^3 \oplus \mathbb{Z} \cdot u^2 x$$

and so on, so the equivariant cohomology of S^2 under rotation is

$$H^\bullet_{S^1}(S^2) \simeq \mathbb{Z}[u] \oplus \mathbb{Z}[u]x, \quad |x| = |u| = 2.$$

Figuring the multiplicative structure is more involved. Section 26.2 in Tu 2020 shows that

$$H^\bullet_{S^1}(S^2; \mathbb{R}) = \frac{\mathbb{R}[u, x]}{(x^2 - u^2)}, |u| = |x| = 2.$$

Borel cohomology is a generalized cohomology theory.

Proposition 5.25. *Borel equivariant cohomology H_G is a generalized cohomology theory, in the sense that it satisfies the following Eilenberg-Steenrod axioms: homotopy invariance, excision, additivity, and exactness.*

A generalized cohomology is not required to satisfy the *dimension axiom*, i.e. $H^\bullet$ of a point is $\mathbb{Z}$. Therefore, we need to determine its value on a single point.

Proposition 5.26. *Let $H^\bullet_G(-; \mathbb{Z})$ be the equivariant cohomology functor for group G. Then,*

$$H^\bullet_G(*; \mathbb{Z}) \cong H^\bullet(BG; \mathbb{Z}).$$

Proof

By definition,

$$\begin{aligned} H^\bullet_G(*;\mathbb{Z}) &\cong H^\bullet(EG \times_G *;\mathbb{Z}) \\ &\cong H^\bullet(EG/G;\mathbb{Z}) \\ &\cong H^\bullet(BG;\mathbb{Z}). \end{aligned}$$

■

Proposition 5.27. *For any G-space M, $H^\bullet_G(M)$ is an $H^\bullet(BG)$-algebra.*

Proof

By functoriality, the constant map $M \to *$ induces an opposite map $H_G(*) \to H_G(M)$, endowing the latter the structure of a module. ■

Example 5.28. Toric equivariant cohomology. Let $G = T \cong (\mathbb{S}^1)^k$ be a torus. To determine $H^\bullet(BT;\mathbb{Z})$, we compute

$$\begin{aligned} BT &= B(\mathbb{S}^1)^k \\ &= (B\mathbb{S}^1)^k \\ &= (\mathbb{CP}^\infty)^k. \end{aligned}$$

By Künneth theorem (in this case, Künneth isomorphism),

$$\begin{aligned} H^\bullet(BT;\mathbb{Z}) &= H^\bullet((\mathbb{CP}^\infty)^k;\mathbb{Z}) \\ &= H^\bullet(\mathbb{CP}^\infty;\mathbb{Z}) \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} H^\bullet(\mathbb{CP}^\infty;\mathbb{Z}) \\ &= \mathbb{Z}[u_1] \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} \mathbb{Z}[u_k] \\ &= \mathbb{Z}[u_1, \ldots, u_k]! \end{aligned}$$

Thus, for any T-space X, $H^\bullet_T(X;\mathbb{Z})$ is an algebra over the polynomial ring

$\mathbb{Z}[u_1, \ldots, u_k]$.

For a detailed treatment of toric equivariant cohomology of Grassmannians, see Chapter 9 of Anderson and Fulton 2023.

5.4 Equivariant de Rham Theory

Every fact about ordinary cohomology and integration of classes has an exact analogue in the equivariant world.

Definition 5.29. For a Lie group G, let $\mathfrak{g}$ be its Lie algebra and $\mathfrak{g}^\vee$ be its dual. Let $\mathrm{Sym}(\mathfrak{g}^\vee)$ be the symmetric algebra over $\mathfrak{g}^\vee$. Then, define the **Cartan complex** of **equivariant differential forms** to be

$$\Omega_G(X) := (\mathrm{Sym}(\mathfrak{g}^\vee) \otimes \Omega_X)^G.$$

The **Cartan differential**

$$D : (\mathrm{Sym}\,(\mathfrak{g}^\vee) \otimes \Omega_X)^G \to (\mathrm{Sym}\,(\mathfrak{g}^\vee) \otimes \Omega_X)^G$$

is given by

$$(D\alpha)(-) = d(\alpha(-)) - \iota_{(-)}(\alpha(-))$$

for $\alpha \in (\mathrm{Sym}(\mathfrak{g}^\vee) \otimes \Omega_X)^G$, where ι is the interior product.

The Cartan complex is a differential model for H_G, because the de Rham isomorphism holds.

Theorem 5.30 (Equivariant de Rham Isomorphism)**.** *For a compact connected Lie group G with Lie algebra $\mathfrak{g}$ and a G-manifold X, there is a graded-algebra isomorphism between equivariant cohomology and the cohomology of the Cartan model:*

$$H^*_G(X) \simeq H^* \left\{ (\mathrm{Sym}\,(\mathfrak{g}^\vee) \otimes \Omega_X)^G , D \right\}.$$

Similarly, one can show that the theories of vector bundles and characteristic classes transfer to the equivariant setting.

Having a well-defined theory of equivariant integration, we would like to translate every ordinary integral into its equivariant counterpart. We achieve that via the following commutative diagrams. Whenever X is compact and oriented, the left diagram of obvious inclusions and projections induces the right diagram on the level of cohomology:

$$\begin{tikzcd} X \arrow[r, "q"] \arrow[d, "\iota"'] \arrow[dr, phantom, "\lrcorner", very near start] & * \arrow[d, "b"] \\ X//G = EG \times_G X \arrow[r, "p"'] & BG \end{tikzcd} \Longrightarrow \begin{tikzcd} H^\bullet(X) & \mathbb{Z} \arrow[l, "q^*"'] \\ H^\bullet_G(X) \arrow[u, "\iota^*"] & H^\bullet(BG) \arrow[l, "p^*"] \arrow[u, "b^*"'] \end{tikzcd}$$

Given such, if we wanted to compute an ordinary integral

$$\int_X \alpha = q_* \alpha \in \mathbb{Z}$$

then we could proceed as follows: suppose we can pick a lift $\tilde{\alpha}$ of α along ι^*. Then one can compute the above integral as

$$b^* p_* \tilde{\alpha},$$

which becomes an *equivariant integral*, now taking values in the polynomial ring $H^\bullet(BG)$! In AG, if variety X is proper over $\operatorname{Spec}\mathbf{k}$, then he map q is well-defined, so the same computation follows through.

Remark 5.31. Notice that equivariant integrals output values in a polynomial ring, instead of giving us numbers. However, magical cancellations usually take place, and we do end up with a number in the end.

The algebraic geometers in the room are surely feeling worried at this point, since the constructions for equivariant cohomology so far relies heavily on the infinite colimit EG which is non-algebraic. Thankfully, as with many other things, we can use a series of algebraic spaces to approximate the role of EG.

Theorem 5.32 (**Approximation Spaces,** Ricolfi 2022 Theorem 7.3.1)**.** *Let $(E_m)_{m\geqslant 0}$ be a family of connected spaces on which G acts freely on the right. Let $\nu : \mathbb{N} \to \mathbb{N}$ be a function such that $\pi_i(E_m) = 0$ for $0 < i < \nu(m)$ and such that $\lim_{m\to\infty} \nu(m) = \infty$. Then, for any left G-action on a space X, there are natural isomorphisms*

$$\mathrm{H}^i_G(X) \cong \mathrm{H}^i\left(E_m \times_G X\right), \quad i < \nu(m).$$

Proof

Proof by induction. Set $E = EG$. The diagonal action of G on $E \times E_m$ induces a commutative diagram

$$\begin{tikzcd}
E_m \times X \arrow[d] & E \times E_m \times X \arrow[l] \arrow[r] \arrow[d] & E \times X \arrow[d] \\
E_m \times_G X & (E \times E_m) \times_G X \arrow[l] \arrow[r] & E \times_G X
\end{tikzcd}$$

where the vertical maps are the (free) quotient maps, and the horizontal maps are locally trivial fibre bundles with fibre indicated on top of the corresponding arrow. As a consequence of the Leray-Hirsch Lemma, if a fibre bundle $Y \to B$ has H^i (fibre) $= 0$ for $0 < i \leqslant m$, then the pullback induces a natural isomorphism $\mathrm{H}^i(B) \xrightarrow{\simeq} \mathrm{H}^i(Y)$ for $i \leqslant m$. (See also Theorem 7.2.15). Since $\nu(m)$ goes to infinity as m grows, we can apply the previous statement to $\nu(m)$ directly, showing that for all $i < \nu(m)$ we have isomorphisms

$$\mathrm{H}^i\left(E_m \times_G X\right) \xrightarrow{\sim} \mathrm{H}^i\left((E \times E_m) \times_G X\right) \xleftarrow{\sim} \mathrm{H}^i_G(X)$$

induced by the lower row of the diagram. ■

In fact, these approximation spaces enables us to extend intersection theory to the equivariant setting. See the following exposition by Edidin and Graham, which appears as arXiv:alg-geom/9609018: https://arxiv.org/abs/alg-geom/9609018

5.5 Atiyah-Bott Localization Formula

Localization formulae are the payoff for developing this equivariant theory. In short, they allow us to compute the integrals of cohomology classes solely on the fixed loci of torus actions.

Theorem 5.33 (Atiyah-Bott Localization Formula). *Denote $H_T^\bullet(*)$ by $\mathcal{H}_T$. Let X be a compact smooth manifold equipped with an action of a torus* T. *Then the equivariant pushforward along $\iota : X^T \hookrightarrow X$ induces an isomorphism*

$$\iota_* : \mathrm{H}_T^*\left(X^T\right) \otimes_{\mathrm{H}_\mathrm{T}^*} \mathcal{H}_T \xrightarrow{\sim} \mathrm{H}_T^*(X) \otimes_{\mathrm{H}_\mathrm{T}^*} \mathcal{H}_\mathrm{T}.$$

Its inverse is given by

$$\psi \mapsto \sum_{\alpha \in X^T} \frac{\iota_\alpha^* \psi}{e^T\left(N_\alpha\right)}.$$

In integral form, we have us, for any equivariant class $\psi \in \mathrm{H}_\mathrm{T}^*(X) \otimes_{\mathrm{H}_\mathrm{T}^*} \mathcal{H}_\mathrm{T}$,

$$\int_X \psi = \sum_\alpha q_{\alpha *} \frac{\iota_\alpha^* \psi}{e^T\left(N_\alpha\right)} = \sum_\alpha \int_{F_\alpha} \frac{\iota_\alpha^* \psi}{e^T\left(N_\alpha\right)} \in \mathcal{H}_T.$$

In particular, if X be a smooth complex projective T–variety with finitely many fixed points. Then for all $\psi \in \mathrm{H}_T^*(X)$ there is an identity

$$\int_X \psi = \sum_{q \in X^\mathrm{T}} \frac{i_q^* \psi}{e^T\left(T_q X\right)} \in \mathcal{H}_T.$$

5.6 27 Lines via Equivariant Localization

cf. Ricolfi 2022. With the knowledge of the localization formula, let's redo the computation

$$\int_{\mathbb{G}(1,3)} c_4(\mathrm{Sym}^3 \mathcal{S}^\vee).$$

Both $\mathbb{P}^3$ and $\mathbb{G}(1,3)$ are toric varieties, with torus action inherited from the $(\mathbb{C}^\times)^4$ action on $\mathbb{C}^4$. Let us assume that the **weights** of this torus action are $(w_0, \ldots, w_3)$, meaning $\mathbf{t} \cdot x_i = t^{w_i} x_i$ for $\mathbf{t} \in T$.

The torus action on $\mathbb{P}^3$ has four fixed points

$$p_0, \ldots, p_3 = (1:0:0:0), \ldots, (0:0:0:1),$$

with 6 fixed lines

$$l_{01}, l_{02}, ..., l_{23}$$

connecting every pair of points. This is reflected by the toric polytope of $\mathbb{P}^3$:

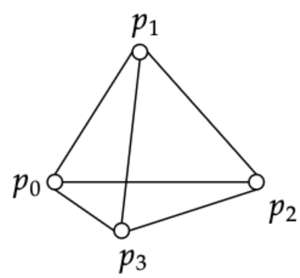


Figure 5.34: Toric polytope of $\mathbb{P}^3$.

Each of the lines l_{ij} are then a toric fixed point on $\mathbb{G}(1,3)$. This is reflected by the moment graph of $\mathbb{G}(1,3)$:

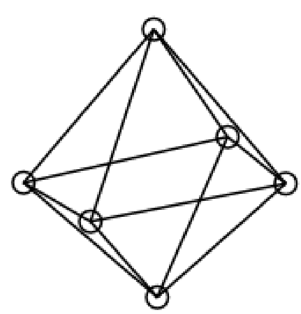


Figure 5.35: Moment graph of $\mathbb{G}(1,3)$.

Now, to compute the desired integral, we may push it forward and compute instead

$$\int_{\mathbb{G}(1,3)} e^T(\mathrm{Sym}^3 \mathcal{S}^\vee),$$

for which we may use the localization formula:

$$\int_{\mathbb{G}(1,3)} e^T(\mathrm{Sym}^3 \mathcal{S}^\vee) = \sum_{\ell_{ij}} \frac{e^{\mathrm{T}}\left(\mathrm{Sym}^3\mathcal{S}^\vee\right)\Big|_{\ell_{ij}}}{e^{\mathrm{T}}\left(T_{\ell_{ij}}\mathbb{G}(1,3)\right)}.$$

To calculate the equivariant Euler class of $T_{l_{ij}}\mathbb{G}(1,3)$, we restrict the tautological sequence

$$0 \to \mathcal{S} \to \mathcal{O}_G \otimes V \to \mathcal{Q} \to 0$$

to the point $l_{ij} \in \mathbb{G}(1,3)$ (and assume the two indices other than i, j are h, k), where it reads

$$0 \to \mathbb{C} \cdot \{x_i, x_j\} \to V \to \mathbb{C} \cdot \{x_h, x_k\} \to 0.$$

Then, the action of T has weights w_i, w_j on $l_{ij} = \mathcal{S}_{l_{ij}}$, and weights w_h, w_k on $l_{ij}^\perp = \mathcal{Q}_{l_{ij}}$. Then, via the identification

$$T_{l_{ij}}G \cong \mathrm{Hom}(\mathcal{S}_{l_{ij}}, \mathcal{Q}_{l_{ij}}) \cong \mathcal{S}_{ij}^\vee \otimes \mathcal{Q}_{l_{ij}},$$

we see that

$$T_{l_{ij}}G \cong \mathbb{C} \cdot \{x_i \otimes x_h^\vee, x_i \otimes x_k^\vee, x_j \otimes x_h^\vee, x_j \otimes x_k^\vee\},$$

whence

$$e^T(T_{l_{ij}}G) = (w_i - w_k)(w_j - w_k)(w_i - w_h)(w_j - w_h) \in \mathcal{H}_T.$$

On the other hand, we have

$$\begin{aligned}\operatorname{Sym}^3 \mathcal{S}_{l_{ij}}^\vee &= \operatorname{Sym}^3 \mathbb{C} \cdot \{x_i, x_j\} \\ &= \mathbb{C} \cdot \{x_i^3, x_i^2 x_j, x_i x_j^2, x_j^3\},\end{aligned}$$

which has torus weights $w_i^3, w_i^2 w_j, w_i w_j^2, w_j^3$, so

$$e^T\left(\operatorname{Sym}^3 \mathcal{S}^\vee\right)\Big|_{l_{ij}} = (3w_i)\left(2w_i + w_j\right)\left(w_i + 2w_j\right)\left(3w_j\right) \in \mathcal{H}_T.$$

Putting it all together, we have

$$\begin{aligned}\sum_{0\leqslant i<j\leqslant 3} \frac{e^T\left(\operatorname{Sym}^3 \mathcal{S}^\vee\right) \mid l_{ij}}{e^{\mathbb{T}}\left(T_{l_{ij}} \mathbb{G}(1,3)\right)} &= \sum_{0\leqslant i<j\leqslant 3} \frac{(3w_i)\left(2w_i + w_j\right)\left(w_i + 2w_j\right)\left(3w_j\right)}{(w_i - w_h)\left(w_j - w_h\right)(w_i - w_k)\left(w_j - w_k\right)} \\ &= 9\frac{w_0\,(2w_0 + w_1)\,(w_0 + 2w_1)\,w_1}{(w_0 - w_2)\,(w_0 - w_3)\,(w_1 - w_2)\,(w_1 - w_3)} \\ &+ 9\frac{w_0\,(2w_0 + w_2)\,(w_0 + 2w_2)\,w_2}{(w_0 - w_1)\,(w_0 - w_3)\,(w_2 - w_1)\,(w_2 - w_3)} \\ &+ 9\frac{w_0\,(2w_0 + w_3)\,(w_0 + 2w_3)\,w_3}{(w_0 - w_1)\,(w_0 - w_2)\,(w_3 - w_1)\,(w_3 - w_2)} \\ &+ 9\frac{w_1\,(2w_1 + w_2)\,(w_1 + 2w_2)\,w_2}{(w_1 - w_0)\,(w_1 - w_3)\,(w_2 - w_0)\,(w_2 - w_3)} \\ &+ 9\frac{w_1\,(2w_1 + w_3)\,(w_1 + 2w_3)\,w_3}{(w_1 - w_0)\,(w_1 - w_2)\,(w_3 - w_0)\,(w_3 - w_2)} \\ &+ 9\frac{w_2\,(2w_2 + w_3)\,(w_2 + 2w_3)\,w_3}{(w_2 - w_0)\,(w_2 - w_1)\,(w_3 - w_0)\,(w_3 - w_1)}.\end{aligned}$$

One can check by any computer algebra system or simply plugging in enough distinct values of $w_0, \ldots, w_3$ that the above rational function is always equal to 27!

As an example, evaluating at $(w_0, \ldots, w_3) = (-1, 0, 1, 2)$, we have

$$\# \text{ Lines on } S = 9(0 - \frac{1}{3} + 0 + 0 + 0 + \frac{10}{3}) = 27!$$

6 Variation ∞: Grothendieck-Riemann-Roch

cf. Eisenbud and Harris 2016. The existence of nice machines like Schubert calculus and equivariant localization relies on the fact that Grassmannians are particularly nice parameter spaces. However, these parameter spaces, being instances of Hilbert schemes, are in essence the representing objects of certain functors. That we know the geometry and combinatorics of them is arguably a pure serendipity.

One may ask, what if no mathematician in the world has ever come up with those nice gadgets? In that case, what might be the last glimmer of hope? Well, we remember that the universal family of lines over G is the incidence correspondence

$$\Phi = \left\{(L, p) \in G \times \mathbb{P}^3 \mid p \in L\right\}$$

we will let $\alpha : \Phi \to G$ and $\beta : \Phi \to \mathbb{P}^3$ be the projection maps. The bundle $\mathcal{E}$ is the direct image of $\mathcal{L} = \beta^*\mathcal{O}_{\mathbb{P}^3}(3)$.

$$\begin{tikzcd}
 & G \times \mathbb{P}^3 \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\
G & & \mathbb{P}^3 \\
\mathcal{E} = \alpha_*\beta^*\mathcal{O}_{\mathbb{P}^3}(3) \arrow[rr, dashed, no head] & & \mathcal{O}_{\mathbb{P}^3}(3)
\end{tikzcd}$$

And our goal was to compute $c_4(\operatorname{Sym}^3 \mathcal{E})$. Namely, we want to compute Chern classes of a *direct image sheaf* from scratch. This is done via the **Grothendieck-Riemann-Roch** theorem.

6.1 The Chern Character

So far, we have used total Chern classes for most computations, but it often feels a little clumsy, since the total Chern class is multiplicative on short exact sequences, and the tensor product formulae are unwieldy. According to Eisenbud & Harris, Hirzebruch in the 60s discovered a ring homomorphism from the 'ring of vector bundles' to the Chow group, called the *Chern character*.

First off, by the 'ring of vector bundles' we mean the K-theory ring $K_0(X)$. For a variety X, the set of finite rank vector bundles $\mathrm{Vect}(X)$ forms a commutative monoid under $\oplus$ and $\otimes$, i.e. it satisfies all axioms of a commutative unital ring, except for the fact that vector bundles do not have additive inverses. Thus, we may form the *group completion* of $\mathrm{Vect}(X)$ by formally adjoining all differences $\mathcal{E} \ominus \mathcal{F}$, termed *virtual vector bundles*.

Definition 6.1. For a variety X, the K-theory ring $K_0(X)$ is the group completion of the commutative monoid $\mathrm{Vect}(X)$.

For a smooth quasiprojective variety X, $A(X)$ is also a commutative unital ring, and the Chern character magically defines a ring homomorphism between $K_0(X)$ and $A(X)$ (more precisely, the rationalization of the two).

Definition 6.2 (Chern character). Let $\mathcal{E}$ be a vector bundle. Using the splitting principle, we may write $c(\mathcal{E}) = \prod(1 + \alpha_i)$. Then, we define the Chern character to be

$$\mathrm{Ch}(\mathcal{E}) = \sum e^{\alpha_i}.$$

In other words, the k-th graded piece $\mathrm{Ch}_k(\mathcal{E})$ of the Chern character is

$$\mathrm{Ch}_k(\mathcal{E}) = \sum \frac{\alpha_i^k}{k!}$$

expressed as a polynomial in the elementary symmetric functions of the α_i and applied to the Chern classes $c_i(\mathcal{E})$.

Example 6.3. The first few cases are

$$\begin{aligned}
\mathrm{Ch}_0(\mathcal{E}) &= \mathrm{rank}(\mathcal{E}), \\
\mathrm{Ch}_1(\mathcal{E}) &= c_1(\mathcal{E}), \\
\mathrm{Ch}_2(\mathcal{E}) &= \frac{c_1(\mathcal{E})^2 - 2c_2(\mathcal{E})}{2}.
\end{aligned}$$

Proposition 6.4. *If X is a smooth quasiprojective variety, then the map*

$$Ch \ : K_0(X) \otimes \mathbb{Q} \to A(X) \otimes \mathbb{Q}$$

is a ring homomorphism.

In fact, for projective varieties, the following wondrous statement is true.

Theorem 6.5 (Grothendieck)**.** *If X is a smooth projective variety, then the map*

$$Ch \ : K_0(X) \otimes \mathbb{Q} \to A(X) \otimes \mathbb{Q}$$

is an isomorphism of rings.

6.2 The Todd Class

Definition 6.6. Suppose $\mathcal{E}$ is a vector bundle/locally free sheaf of rank n on a smooth variety X. We formally factor its Chern class by the splitting principle:

$$c(\mathcal{E}) = \prod_{i=1}^{n} (1 + \alpha_i) .$$

We define the **Todd class** of $\mathcal{E}$ to be

$$\mathrm{Td}(\mathcal{E}) = \prod_{i=1}^{n} \frac{\alpha_i}{1 - e^{-\alpha_i}},$$

written as a power series in the elementary symmetric polynomials $c_i(\mathcal{E})$ of the α_i.

Example 6.7. To calculate the first few terms of the Todd class, write so

$$\begin{aligned} 1 - e^{-\alpha} &= \alpha - \frac{\alpha^2}{2} + \frac{\alpha^3}{6} - \frac{\alpha^4}{24} + \cdots , \\ \frac{1 - e^{-\alpha}}{\alpha} &= 1 - \frac{\alpha}{2} + \frac{\alpha^2}{6} - \frac{\alpha^3}{24} + \frac{\alpha^4}{120} - \cdots ; \end{aligned}$$

inverting this, we get

$$\frac{\alpha}{1 - e^{-\alpha}} = 1 + \frac{\alpha}{2} + \frac{\alpha^2}{12} - \frac{\alpha^4}{720} + \cdots ,$$

so

$$\mathrm{Td}(\mathcal{E}) = \prod_{i=1}^{n} \left(1 + \frac{\alpha_i}{2} + \frac{\alpha_i^2}{12} - \frac{\alpha_i^4}{720} + \cdots \right) .$$

Rewriting the first few of these in terms of the symmetric polynomials of the α_i-that is, the Chern classes of $\mathcal{E}$-we get formulas for the first few terms of the

Todd class:

$$\begin{aligned}
\mathrm{Td}_0(\mathcal{E}) &= 1 \\
\mathrm{Td}_1(\mathcal{E}) &= \sum \frac{\alpha_i}{2} = \frac{c_1(\mathcal{E})}{2} \\
\mathrm{Td}_2(\mathcal{E}) &= \frac{1}{12}\sum \alpha_i^2 + \frac{1}{4}\sum_{i<j}\alpha_i\alpha_j = \frac{c_1^2(\mathcal{E}) + c_2(\mathcal{E})}{12} \\
\mathrm{Td}_3(\mathcal{E}) &= \frac{1}{24}\sum_{i\neq j}\alpha_i\alpha_j^2 = \frac{c_1(\mathcal{E})c_2(\mathcal{E})}{24}
\end{aligned}$$

Like the total Chern class, the Todd class is also multiplicative on SES:

Proposition 6.8. *For a short exact sequence of vector bundles*

$$0 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}'' \longrightarrow 0,$$

the Todd classes satisfy

$$\mathrm{Td}(\mathcal{E}) = \mathrm{Td}(\mathcal{E}')\,\mathrm{Td}(\mathcal{E}'').$$

The reason Todd defined these classes is that they are generalizations of the Euler characteristic.

Proposition 6.9. *Let X be a smooth n-dimensional variety. Then, the Euler characteristic is recovered as the nth graded piece of the Todd class:*

$$\chi\left(\mathcal{O}_X\right) = \deg \mathrm{Td}_n\left(\mathcal{T}_X\right)$$

6.3 Grothendieck-Riemann-Roch

We now have the necessary technology to state Grothendieck-Riemann-Roch.

Theorem 6.10 (**Grothendieck-Riemann-Roch**). *Consider a proper morphism $f : X \to Y$ between smooth quasi-projective schemes and a bounded complex of sheaves $\mathcal{F}^\bullet$ on X. We can form the K-theoretic pushforward*

$$f_! = \sum (-1)^i R^i f_* : K_0(X) \to K_0(Y)$$

(alternating sum of higher direct images) and the proper pushforward

$$f_* : A(X) \to A(Y).$$

Then, the following formula holds:

$$\mathrm{Ch}\left(f_! \mathcal{F}^\bullet\right) \mathrm{Td}(Y) = f_* \left(\mathrm{Ch}\left(\mathcal{F}^\bullet\right) \mathrm{Td}(X)\right).$$

In other words, the following diagram commutes.

$$\begin{tikzcd}
K_0(X) \arrow[r, "Ch"] \arrow[d, "f_!"'] & A(X)_{\mathbb{Q}} \arrow[d, "f_*"] \arrow[loop above, "Td(X)"] \\
K_0(Y) \arrow[r, "Ch"'] & A(Y)_{\mathbb{Q}} \arrow[loop below, "Td(Y)"']
\end{tikzcd}$$

The formula may be thought of as a precise measure of 'lack of commutativity' of taking pushforwards, where the correction terms are given by the Todd classes. The 'classical' Hirzebruch-Riemann-Roch formula is then a specialization of G-R-R, with $Y = pt$ and $\mathbf{k} = C$.

Corollary 6.11 (Hirzebruch-Riemann-Roch). *If X is a smooth projective variety of dimension n and $\mathcal{F}$ a coherent sheaf on X, then*

$$\chi(\mathcal{F}) = \deg \mathrm{Ch}(\mathcal{F}) \, \mathrm{Td}\left(\mathcal{T}_X\right).$$

We may specialize further to the cases of curves and surfaces, as one typically sees in an introductory course on algebraic geometry:

Corollary 6.12 (R-R for surfaces)**.** *If $\mathcal{F}$ is a coherent sheaf on a smooth projective surface S, then*

$$\chi(\mathcal{F}) = \frac{c_1(\mathcal{F})^2 - 2c_2(\mathcal{F}) + c_1(\mathcal{F})c_1\left(\mathcal{T}_C\right)}{2} + \mathrm{rank}(\mathcal{F})\frac{c_1\left(\mathcal{T}_S\right)^2 + c_2\left(\mathcal{T}_S\right)}{12}$$

Corollary 6.13 (R-R for curves)**.** *If $\mathcal{F}$ is a coherent sheaf on a smooth curve C, then*

$$\chi(\mathcal{F}) = c_1(\mathcal{F}) + \mathrm{rank}(\mathcal{F})\frac{c_1\left(\mathcal{T}_C\right)}{2}$$

Proof

If we take $X = C$ and $Y = \{*\}$ a point, then the Grothendieck-Riemann-Roch formula reads as

$$\begin{aligned}
\mathrm{ch}\left(f_! E\right) &= h^0(C, E) - h^1(C, E) \\
f_*(\mathrm{ch}(E)\,\mathrm{td}(X)) &= f_*\left(\left(n + c_1(E)\right)\left(1 + (1/2)c_1\left(T_C\right)\right)\right) \\
&= f_*\left(n + c_1(E) + (n/2)c_1\left(T_C\right)\right) \\
&= f_*\left(c_1(E) + (n/2)c_1\left(T_C\right)\right) \\
&= d + n(1-g)
\end{aligned}$$

hence,

$$\chi(C, E) = d + n(1-g).$$

$\blacksquare$

6.4 27 Lines via G-R-R

Now, let's try to use the overpowered G-R-R machine to calculate the number of lines on a smooth cubic surface. Recall that

$$\mathcal{E} = \operatorname{Sym}^3 \mathcal{S}^\vee.$$

To compute the Chern classes of $\mathcal{E}$ we first observe that the restriction of $\mathcal{O}_L(3)$ of $\mathcal{L} = \beta^* \mathcal{O}_{\mathbb{P}^n}(3)$ to each fiber $\Phi_{[L]} = \alpha^{-1}([L]) = L \cong \mathbb{P}^1$ is $\mathcal{O}_{\mathbb{P}^1}(3)$, which has no higher cohomology. From the theorem on cohomology and base change (Theorem B.5), it follows that the direct image

$$\mathcal{E} = \alpha_* \mathcal{L} = \alpha_* \left(\beta^* \mathcal{O}_{\mathbb{P}^3}(3)\right)$$

is locally free, with fiber $H^0\left(\mathcal{O}_L(3)\right)$ at $[L]$. Because of the vanishing of the higher cohomology of $\mathcal{L}$ on the fiber of α, the higher direct images $R^i \alpha_*(\mathcal{L})$ are 0 for $i > 0$, so the Grothendieck Riemann-Roch theorem becomes a formula for the Chern character of $\mathcal{E}$:

$$\operatorname{Ch}(\mathcal{E}) = \alpha_* \left(\operatorname{Ch}(\mathcal{L}) \cdot \operatorname{Td}\left(\mathcal{T}^v_{\Phi/G}\right)\right).$$

To evaluate this explicitly requires the following steps:

(a) Describe the Chow ring $A(\Phi)$.

(b) Describe the direct image map $\alpha_* : A(\Phi) \to A(G)$.

(c) Calculate the Chern character of $\mathcal{L}$ and the Todd class of the relative tangent bundle $\mathcal{T}^v_{\Phi/G}$.

(d) Take the direct image of their product, to arrive at $\mathrm{Ch}(\mathcal{E})$.

(e) Finally, convert this back into the Chern classes of $\mathcal{E}$.

(a) We notice that $\Phi = \mathbb{P}\mathcal{S}$ is the projectivization of the tautological bundle. We have

Proposition 6.14 (E & H Proposition 9.10)**.** *Let $G = \mathbb{G}(k, n)$ be the Grassmannian of k-planes in $\mathbb{P}^n$ and $\Phi \subset G \times \mathbb{P}^n$ the universal k-plane as above, with $\pi : \Phi \to G$ and $\eta : \Phi \to \mathbb{P}^n$ the projection maps. We have then*

$$A(\Phi) = A(G)[\zeta]/\left(\zeta^{k+1} - \sigma_1\zeta^k + \sigma_{1,1}\zeta^{k-1} + \cdots + (-1)^{k+1}\sigma_{1,1,\ldots,1}\right)$$

where $\zeta \in A^1(\Phi)$ is the tautological class, or equivalently the pullback via η of the hyperplane class in $\mathbb{P}^n$.

By the above,

$$A(\Phi) = A(G)[\zeta]/\left(\zeta^2 - \sigma_1\zeta + \sigma_{1,1}\right)$$

where ζ is the hyperplane class of $\mathbb{P}^n$.

(b) let s denote the total Segre class (the formal power series inverse to the total Chern class). We have

$$\alpha_*\left(1 + \zeta + \zeta^2 + \cdots\right) = s(\mathcal{S}) = \frac{1}{c(\mathcal{S})} = \frac{1}{1 - \sigma_1 + \sigma_{1,1}} = 1 + \sigma_1 + \sigma_2.$$

In other words,

$$\alpha_*\zeta = 1, \quad \alpha_*(\zeta^2) = \sigma_1, \quad \alpha_*(\zeta^3) = \sigma_2 \quad \text{and} \quad \alpha_*(\zeta^4) = 0.$$

(c) To compute the Chern character of $\mathcal{L}$, we first observe that, since the fiber of the line bundle $\mathcal{O}_{\mathbb{P}\mathcal{S}}(1)$ at a point $(L, p) \in \Phi$ is the dual of the one-dimensional vector subspace of $\mathbb{C}^4$ corresponding to p, we have

$$\zeta = c_1\left(\mathcal{O}_{\mathbb{P}\mathcal{S}}(1)\right) = \beta^* c_1\left(\mathcal{O}_{\mathbb{P}^3}(1)\right)$$

In particular, it follows that

$$c_1(\mathcal{L}) = 3\zeta$$

and so

$$\mathrm{Ch}(\mathcal{L}) = 1 + 3\zeta + \frac{9}{2}\zeta^2 + \frac{27}{6}\zeta^3,$$

since higher powers of ζ vanish.

For the Todd class of the relative tangent bundle, if we denote by $\mathcal{U}$ the tautological line bundle on $\Phi = \mathbb{P}\mathcal{S}$, and by $\mathcal{Q}$ the tautological quotient bundle, we have

$$\mathcal{T}^v_{\Phi/G} = \mathcal{U}^* \otimes \mathcal{Q}$$

From the exact sequence

$$0 \longrightarrow \mathcal{U} \longrightarrow \alpha^*\mathcal{S} \longrightarrow \mathcal{Q} \longrightarrow 0$$

we see that

$$c_1(\mathcal{Q}) = c_1(\alpha^*\mathcal{S}) - c_1(\mathcal{U}) = -\sigma_1 + \zeta$$

and hence

$$c_1\left(\mathcal{T}^v_{\Phi/G}\right) = c_1(\mathcal{U}^* \otimes \mathcal{Q}) = \zeta + c_1(\mathcal{Q}) = -\sigma_1 + 2\zeta.$$

Plugging this into the formula for the Todd class, we have

$$\mathrm{Td}\left(\mathcal{T}^v_{\Phi/G}\right) = 1 + \frac{2\zeta - \sigma_1}{2} + \frac{(2\zeta - \sigma_1)^2}{12} - \frac{(2\zeta - \sigma_1)^4}{720}.$$

(d) Taking the product of the above Chern character and Todd class, we have

$$\begin{aligned}\mathrm{Ch}(\mathcal{L})\,\mathrm{Td}\left(\mathcal{T}^v_{\Phi/G}\right) = 1 &+ \frac{1}{2}(8\zeta - \sigma_1) + \frac{1}{12}\left(94\zeta^2 - 22\sigma_1\zeta + \sigma_1^2\right)\\ &+ \frac{1}{12}\left(120\zeta^3 - 39\sigma_1\zeta^2 + 3\sigma_1^2\zeta\right)\\ &+ \frac{1}{720}\left(-2668\sigma_1\zeta^3 + 246\sigma_1^2\zeta^2 + 8\sigma_1^3\zeta - \sigma_1^4\right)\\ &+ \frac{1}{720}\left(198\sigma_1^2\zeta^3 + 24\sigma_1^3\zeta^2 - 3\sigma_1^4\zeta\right).\end{aligned}$$

Applying the direct image map found in step (b), we find that by Grothendieck Riemann-Roch

$$\mathrm{Ch}(\mathcal{E}) = 4 + 6\sigma_1 + (7\sigma_2 - 3\sigma_{1,1}) - 3\sigma_{2,1} + \frac{1}{3}\sigma_{2,2}.$$

(e) Finally, it remains to recover the Chern classes from this Chern character. We have

$$c_1(\mathcal{E}) = \mathrm{Ch}_1(\mathcal{E}) = 6\sigma_1$$

and

$$
\begin{aligned}
c_2(\mathcal{E}) &= \frac{1}{2}\mathrm{Ch}_1(\mathcal{E})^2 - \mathrm{Ch}_2(\mathcal{E}) \\
&= 18\sigma_1^2 - (7\sigma_2 - 3\sigma_{1,1}) \\
&= 11\sigma_2 + 21\sigma_{1,1}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
c_3(\mathcal{E}) &= \frac{1}{6}\mathrm{Ch}_1(\mathcal{E})^3 - \mathrm{Ch}_1(\mathcal{E})\mathrm{Ch}_2(\mathcal{E}) + 2\mathrm{Ch}_3(\mathcal{E}) \\
&= 36\sigma_1^3 - 6\sigma_1\,(7\sigma_2 - 3\sigma_{1,1}) - 6\sigma_{2,1} \\
&= 72\sigma_{2,1} - 24\sigma_{2,1} - 6\sigma_{2,1} \\
&= 42\sigma_{2,1},
\end{aligned}
$$

and, finally, the payoff!

$$
\begin{aligned}
c_4(\mathcal{E}) &= \frac{1}{24}\mathrm{Ch}_1(\mathcal{E})^4 - \frac{1}{2}\mathrm{Ch}_1(\mathcal{E})^2\mathrm{Ch}_2(\mathcal{E}) + \frac{1}{2}\mathrm{Ch}_2(\mathcal{E})^2 + 2\mathrm{Ch}_1(\mathcal{E})\mathrm{Ch}_3(\mathcal{E}) - 6\mathrm{Ch}_4(\mathcal{E}) \\
&= 54\sigma_1^4 - 18\sigma_1^2\,(7\sigma_2 - 3\sigma_{1,1}) + \frac{1}{2}\,(7\sigma_2 - 3\sigma_{1,1})^2 - 36\sigma_1\sigma_{2,1} - 2\sigma_{2,2} \\
&= (108 - 72 + 29 - 36 - 2)\sigma_{2,2} \\
&= \mathbf{27}\,\sigma_{2,2}.
\end{aligned}
$$

7 Ending Remarks

We have seen in this document 4 ways of computing the number 27 and 1 way of verifying it. However, this is certainly not the end of the end of the story. One may ask how this count differs as we take $\mathbf{k}$ to be non-algebraically closed, take the cubic surface S to be singular, etc. An emerging field of research is to enrich these counting results using $\mathbb{A}^1$-homotopy theory. See Brazelton 2023.

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