

Intersection Theory Notes 05/31

3264 and All That

‘Michael’ Zeng, Ruofan

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1 Steiner's Conic Problem

For this exposition, fix the ground field $\mathbf{k} = \mathbb{C}$. Recall, a *plane conic* in $\mathbb{P}^2$ is cut out by a degree 2 homogeneous equation

$$a_0X^2 + a_1Y^2 + a_2Z^2 + a_3XY + a_4YZ + a_5ZX = 0,$$

where the zero set is invariant under non-zero scalar multiplication, so the parameter space for plane conics is $\mathbb{P}^5$.

If we fix a conic C_i , then we can describe the locus Z_i in $\mathbb{P}^5$ of conics tangent to C_i by the following incidence correspondence:

$$\Phi = \{(C, P) \in \mathbb{P}^5 \times C \mid C \text{ is tangent to } C_i \text{ at } P.\}$$

where the projection onto the two factors are as follows:

$$\begin{array}{ccc} & \Phi & \\ \pi_{Z_i} \swarrow & & \searrow \pi_{C_i} \\ Z_i & & C_i \end{array}$$

The fiber of the projection π_{C_i} corresponds to conics tangent to C_i at P , i.e. the equation of C restricted to C' should vanish at P to at least order 2, which is a linear condition that cuts dimension by 2. Hence, the fibers of π_{C_i} are linear subspaces of dimension $5 - 2 = 3$. On the other hand, the projection π_{Z_i} is an isomorphism as always, so Z_i is a $\mathbb{P}^3$ -bundle over C_i . We conclude that Z_i is an irreducible hypersurface in $\mathbb{P}^5$. We see that if we intersect 5 of these loci, we should expect that the dimension is cut down to zero, as long as all intersections are *transverse*, and we can then count the number of points in this intersection as

usual. This motivates the following question:

Problem 1.1. (Steiner's Conic Problem) How many plane conics are simultaneously tangent to 5 general conics in $\mathbb{P}^2$?

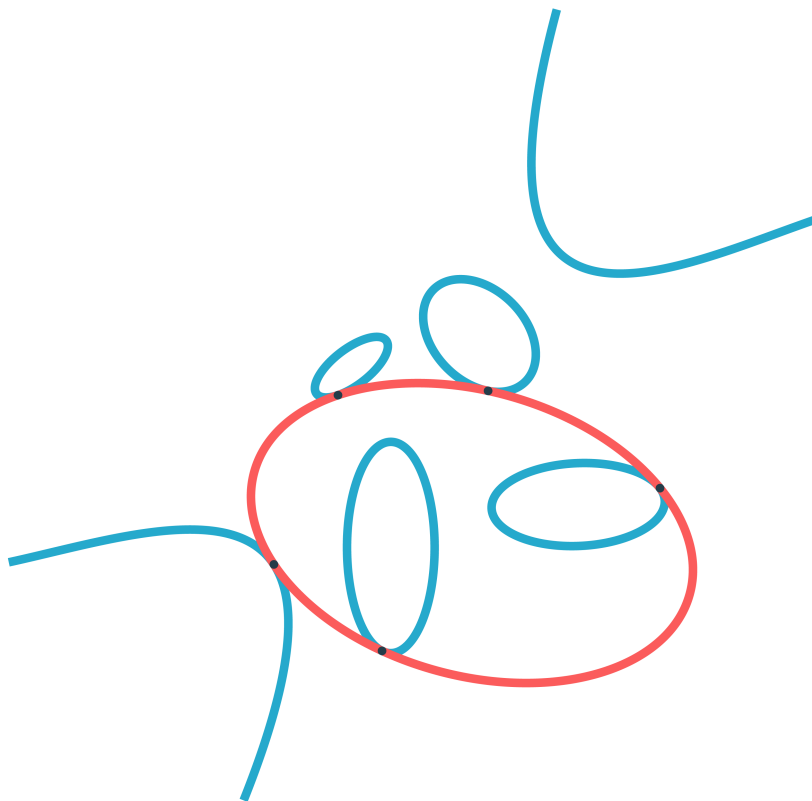


Figure 1.2: The red conic is simultaneously tangent to all 5 given conics. Illustrated by B. S. Paul Breiding and Timme [2019](#)

As we will soon see, this seemingly innocent question requires a great deal of extra work.

1.1 Naïve Attempt

We remember our solution to **circles of Apollonius**, which asks how many plane circles are simultaneous tangent to 3 general circles in $\mathbb{P}^2$. In that calculation, we identified the loci Z_i of circles tangent to a given circle C_i , which was a degree 2 hypersurface in $\mathbb{P}^3$, the parameter space of plane circles. We then used Bezout's theorem and calculated that the number of points in $Z_1 \cap Z_2 \cap Z_3$ was 8 and verified transversality. The five conics problem is apparently a generalization of the circles problem, so we may attempt to run the same arguments.

We have seen that the loci Z_i of conics tangent to a given plane conic C_i is a hypersurface in $\mathbb{P}^5$, so we only need to figure out its degree. To do this, we may repeat our application of *Hurwitz's formula*.

Theorem 1.3 (Riemann-Hurwitz formula cf. Hartshorne IV.2.4, p301.). *Let $f : X \rightarrow Y$ be a finite separable morphism of curves. Let $n = \deg f$ and R be the ramification divisor. Then*

$$2g(X) - 2 = n \cdot (2g(Y) - 2) + \deg R.$$

See Eisenbud & Harris §7.7 for a generalization.

To calculate the degree of Z_i , we need to intersect it with a transverse line (pencil) of conics $\Lambda = \{D_t\}$ in $\mathbb{P}^5$. Say two conics D_0, D_∞ in $\Lambda \setminus Z_i$ has equations F and G . Like before,

$$\#\Lambda \cap Z_i = \# \text{ ramification points of } F/G = \deg R(F/G)$$

where R stands for the ramification divisor, and F/G is viewed as a rational

function from $C_i \cong \mathbb{P}^1$ to $\mathbb{P}^1$.

Now, F is zero on the 4 intersection points of $D_0 \cap C_i$, and G is zero 4 times on $D_\infty \cap C_i$, so F/G is a map of degree 4. Plugging in $n = 4$ and $g(\mathbb{P}^1) = 0$ into the Hurwitz formula, we conclude that

$$\deg Z_i = \deg R(F/G) = 6.$$

So if everything works like before, we should expect that

$$\begin{aligned} \#Z_1 \cap \cdots \cap Z_5 &= 6^5 \\ &= 7776. \end{aligned}$$

In fact, this is the number that Steiner himself in the year 1848 calculated. Unfortunately, this is incorrect!

1.2 Why not 7776?

The failure of the above solution lies in the last step: verifying transversality. The calculation for 7776 holds up as long as the intersections can be made generically transverse. This is our first example where this fails!

Problem 1.4. The loci Z_i can never be made transverse, because every degenerate conic of the double line type is automatically tangent to any given conic.

Indeed, any double line intersects a given conic in $\mathbb{P}^2$, and each point of intersection automatically has multiplicity ≥ 2 . Therefore, each Z_i necessarily contains the singular locus in $\mathbb{P}^5$ of double lines, which traces out a Veronese

surface, so the intersection $Z_1 \cap \cdots \cap Z_5$ is necessarily infinite!

*** picture

Given this knowledge, it is stupid to ask how many possibly degenerate conics are tangent to 5 given ones. Instead, we are forced to modify our enumerative problem to *smooth* conics:

Problem 1.5. (Steiner's Conic Problem, modified) How many **smooth** plane conics are simultaneous tangent to 5 general conics in $\mathbb{P}^2$?

The undesirable locus S of double lines is called the locus of **excess intersections**. There are several ways to remove its contribution to our count:

1. **Blowing up the excess locus.** We may blow up $\mathbb{P}^5$ along S and calculate the intersection $[Z_i]^5$ in $\text{Bl}_S \mathbb{P}^5$. This can be slightly cumbersome as the Chow ring of a blow up is less computationally accessible. Griffith and J. Harris 1994 §6.1 has an account of this method.
2. **Excess intersection formulas.** Originally introduced at the greatest generality in W. Fulton 1984 §7, these formulas provide a direct way of calculating the contribution of each component in the intersection to the number provided by the naive intersection product. Eisenbud and Harris 2016 carries out this method in §13.3.5.
3. **Compactifying the parameter space.** Now that we have limited our attention to *smooth* conics, the corresponding subset in $\mathbb{P}^5$ is no longer a closed locus. However, we may elect to modify our parameter space in such a way that we obtain a closed locus again. The trouble is that there are *many* ways to compactify a parameter space. As Eisenbud & Harris put, finding the right

compactification is closer to an art. In the case of smooth conics, the *space of complete conics* and *Kontsevich spaces* are two of the compactifications that are worth considering.

In the following sections, we attempt to give an account of the second and third methods.

2 Solution 1: the Space of Complete Conics

2.1 Complete Conics

We would like to modify the parameter space in such a way that the smooth locus stays unchanged, but we have better control over degenerate conics (especially double lines). One important idea in finding the right parameter space is to consider *dual conics*. We recall that the parameter space for lines in $\mathbb{P}^2$ is another $\mathbb{P}^2$, and the identification of $\mathbb{P}^2$ with its dual $(\mathbb{P}^2)^\vee$ sends points to lines and lines to points.

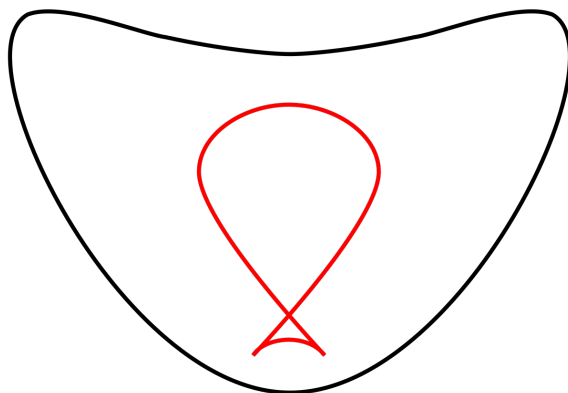


Figure 2.1: An example of dual curves (Wikipedia)

The dual of a conic is another conic, and the dual of a smooth conic is also smooth. Furthermore, the double dual of a conic is itself. Therefore, it is reasonable to consider the following symmetric construction.

Definition 2.2. Let $U = \{(C, C^\vee) \in \mathbb{P}^5 \times \mathbb{P}^{5\vee} \mid C \text{ a smooth conic in } \mathbb{P}^2 \text{ and } C^\vee \subset \mathbb{P}^{2\vee} \text{ its dual}\}$ be the open set of **complete conics**. Then, we define the **variety of complete conics** to be its closure $X = \bar{U} \subset \mathbb{P}^5 \times \mathbb{P}^{5\vee}$.

Since we took the closure, we have necessarily picked up points corresponding to degenerate conics. With a bit of effort, we can make the following classification of complete conics:

1. **Smooth complete conics.** $(C, C') \in U$; that is, C and C' are both smooth and $C' = C^\vee$.
2. **A union of lines and a double line.** $C = L \cup M$, and $C' = 2p^\vee$, where $p^\vee \subset \mathbb{P}^{2\vee}$ is the line dual to $p = L \cap M$.
3. **A double line and a union of lines.** $C = 2L$, and $C' = p^\vee \cup q^\vee$ is the union of the lines in $\mathbb{P}^{2\vee}$ dual to two points $p, q \in L$.
4. **A couple of two double lines.** $C = 2L$, and $C' = 2p^\vee$ is the double of the line in $\mathbb{P}^{2\vee}$ dual to a point $p \in L$.

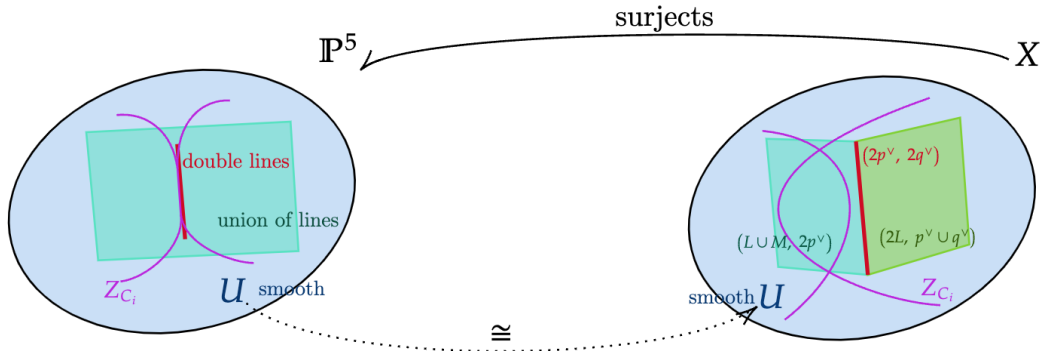


Figure 2.3: Singular loci in the two parameter spaces.

In the next subsection, we verify this classification using a bit of multilinear algebra.

2.2 Degeneration of Complete Conics

Recall that we have the following identifications as long as $\text{char } \mathbf{k} \neq 2$:

- A symmetric linear map $\varphi : V \rightarrow V^\vee$;
- A quadratic form $q : V \rightarrow \mathbf{k}$;
- An element $q' \in \text{Sym}^2 V^\vee$.

Furthermore, a quadric hypersurface Q is defined as the zero locus of q in $\mathbb{P}V$. In particular, a plane conic is defined by a quadratic form on $V = \mathbb{C}^3$.

We see that Q is smooth if and only if the linear map φ is an isomorphism. Q becomes singular whenever φ fails to be an isomorphism, in which case $\text{Sing } Q = \mathbb{P}(\ker \varphi)$. Therefore, it is reasonable to introduce the following notion of rank for Q :

Definition 2.4. The **rank** of a quadric hypersurface Q is defined to be $\text{rk } \varphi$, or equivalently, $\dim \mathbb{P}V - \dim \text{Sing } Q$.

We can explicitly find the equations of X using quadratic forms:

Proposition 2.5. *The variety*

$$X \subset \mathbb{P}(\text{Sym}^2 V^\vee) \times \mathbb{P}(\text{Sym}^2 V) = \mathbb{P}^5 \times \mathbb{P}^{5\vee}$$

of complete conics is smooth and irreducible. Thinking of $(\varphi, \psi) \in \mathbb{P}^5 \times \mathbb{P}^{5\vee}$ as coming from a pair of symmetric matrices $\varphi : V \rightarrow V^\vee$ and $\psi : V^\vee \rightarrow V$, the scheme X is defined by the ideal I generated by the eight bilinear equations specifying that the product $\psi \circ \varphi$ has its diagonal entries equal to one another (two equations) and its off-diagonal entries equal to zero (six equations).

As a consequence, the rank of the quadratic form determines the type of degeneracy of a complete conic:

Corollary 2.6. *If $(\varphi, \psi) \in X$, then one of the following holds:*

- (a) *(Smooth complete conics) If φ is of rank 3, then ψ must be its inverse.*
- (b) *(union of lines and double line) If φ is of rank 2, then (since X is symmetric) the products $\psi \circ \varphi$ and $\varphi \circ \psi$ must both be zero; it follows that ψ is the unique (up to scalars) symmetric map $V^\vee \rightarrow V$ whose kernel is the image of φ and whose image is the kernel of φ .*
- (c) *(double lines and union of lines) If φ is of rank 1, ψ may have rank 1 or 2; in the latter case, it may be any symmetric map $V^\vee \rightarrow V$ whose kernel is the image of φ and whose image is the kernel of φ .*
- (d) *(pair of double lines) If φ and ψ both have rank 1, they simply have to satisfy the condition that the kernel of ψ contains the image of φ and vice versa.*

2.3 Solution 1 to the Five Conics Problem

We have established that the variety X of complete conics is smooth and projective. To redo the calculation of the five conics problem, we consider the locus $Z_{C_i} \subseteq X$ of complete conics $(C, C^\vee)$ where C is tangent to a given conic C_i . We let $\zeta := [Z_{C_i}]$ denote its class in the Chow ring $A(X)$. Our argument requires the following steps:

- (a) **No more excess intersection.** We have to show that the intersection $Z_{C_1} \cap \cdots \cap Z_{C_5}$ contains only *smooth* complete conics.

(b) **Transversality.** We have to show that the intersection $Z_{C_1} \cap \cdots \cap Z_{C_5}$ is transverse.

(c) **Intersection product.** We need to find a way to evaluate the product ζ^5 in $A(X)$.

(a) Complete conics tangent to five conics are smooth. We need to check that the intersection cannot contain each singularity type of complete conics.

For (C, C') equals a union of lines and a double line, it suffices to check the case when C is the union of two lines. This implies the other case by symmetry. So let $(C, C') = (L \cup M, 2p^\vee)$ where $p = L \cap M$. Then, if C is tangent to a given conic C_i , then either L (or M) is a tangent line to C_i , or C_i contains the double point p .

Now suppose that the above (C, C') lies in the intersection $Z_{C_1} \cap \cdots \cap Z_{C_5}$. Since the conics C_i are general, no three conics should be concurrent, so p can be contained in up to 2 of the given conics. We study each of the cases.

(i) *p lies on none of the given conics.* Then, L and M together must be tangent to *all five* conics, which means each of the five conics has to contain one of the dual points $L^\vee$ or $M^\vee$. However, since $C_i^\vee$ are general, $L^\vee$ and $M^\vee$ are contained in a total of at most 4 conics! Therefore, this case cannot happen.

(ii) *p lies on one of the given conics.* Say $p \in C_1$. Since C_1 is general with respect to C_2, C_3, C_4 and C_5 , it will not contain any of the finitely many points of pairwise intersection of lines tangent to two of them. Thus L and M cannot each be tangent to two of the conics $C_2, \dots, C_5$, and once more we see that $(L \cup M, 2p^\vee)$ for at most four of the C_i .

(iii) p lies on two of the given conics. Say $p \in C_1 \cap C_2$. Since C_3, C_4 and C_5 are general with respect to C_1 and C_2 , none of the finitely many lines tangent to two of them passes through a point of $C_1 \cap C_2$; thus L and M can each be tangent to at most one of the conics C_3, C_4 and C_5 , and again we see that $(L \cup M, 2p^\vee)$ is contained in at most four of the C_i .

Hence, the intersection $Z_{C_1} \cap \cdots \cap Z_{C_5}$ cannot contain degenerate complete conics of type $(L \cap M, 2p^\vee)$, and by symmetry, $(2L, p^\vee \cup q^\vee)$.

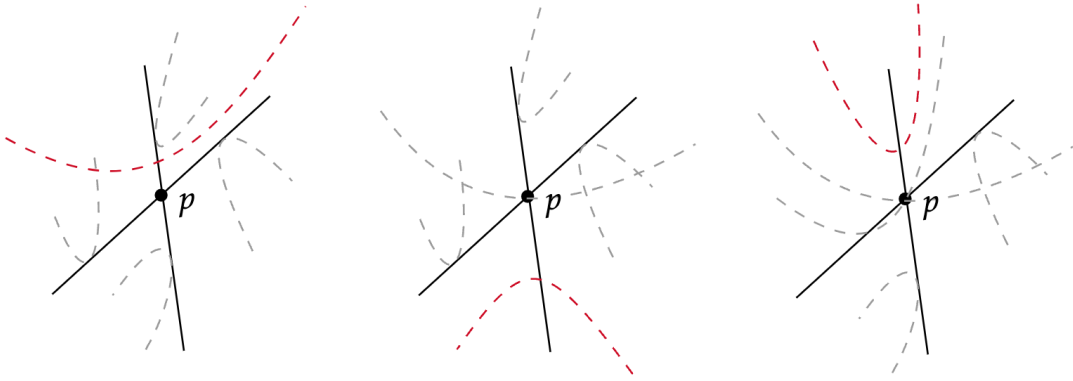


Figure 2.7: In each of the three cases, $(L \cap M, 2p^\vee)$ can be tangent to up to 4 of the conics.

A similar argument shows that this intersection cannot contain $(2L, 2p^\vee)$. We have eliminated excess intersections by constructing the new loci!

(b) Transversality. Since the intersection now only contains smooth complete conics, we may check transversality only on the open set of smooth complete conics, which is then isomorphic to the open set of smooth ordinary conics in $\mathbb{P}^5$. So let $Z_{C_i}^\circ$ be the locus of *smooth* plane conics tangent to the given C_i . Like before,

we try to identify its tangent space with something else. This time, the projective tangent space to $Z_{C_i}^\circ$ at a point p can be identified with the *hyperplane of conics passing through p* .

Lemma 2.8 (Eisenbud and Harris 2016 Proposition 8.6, Tangent Spaces to a Discriminant Hypersurface). *Let $\mathbb{P}^d = \mathbb{P}H^0(\mathcal{O}_{\mathbb{P}^1}(d))$ be the space of polynomials of degree d on $\mathbb{P}^1$ and $\mathcal{D} \subset \mathbb{P}^d$ the discriminant hypersurface, that is, the locus of polynomials with a repeated root. If $F \in \mathcal{D}$ is a point corresponding to a polynomial with exactly one double root p and $d - 2$ simple roots, then $\mathcal{D}$ is smooth at F with tangent space the space of polynomials vanishing at p .*

Proposition 2.9. *Let $C_i \subset \mathbb{P}^2$ be a smooth conic curve and $Z_{C_i}^\circ$ as above. Then, If $C \in Z_{C_i}$ has a point p of simple tangency with C_i (i.e. multiplicity 2) and is otherwise transverse, then $Z_{C_i}^\circ$ is smooth at $[C]$. In this case, the projective tangent plane $\mathbb{T}_{[C]}Z_{C_i}^\circ$ to $Z_{C_i}^\circ$ at $[C]$ is the hyperplane $H_p \subset \mathbb{P}^5$ of conics passing through p .*

Proof

The inclusion

$$\mathbb{P}^1 \cong C_i \rightarrow \mathbb{P}^2$$

induces a restriction of homogeneous forms

$$H^0(\mathcal{O}_{\mathbb{P}^2}(2)) \rightarrow H^0(\mathcal{O}_{C_i}(2)) \cong H^0(\mathcal{O}_{\mathbb{P}^1}(4))$$

which is surjective, with kernel spanned by the equation of C_i . Therefore, if we projectivize, then we obtain a rational map in the form of the projection from the point $[C_i]$:

$$\pi_{[C_i]} : \mathbb{P}^5 \cong \mathbb{P}H^0(\mathcal{O}_{\mathbb{P}^2}(2)) \dashrightarrow \mathbb{P}H^0(\mathcal{O}_{\mathbb{P}^1}(4)) \cong \mathbb{P}^4.$$

The closure $Z_{C_i}^\circ$ in $\mathbb{P}^5$ is thus the cone with vertex $[C_i]$ over the hypersurface $\mathcal{D} \subset \mathbb{P}^4$ of singular divisors in the linear system $|\mathcal{O}_{\mathbb{P}^1}(4)|$, i.e. a discriminant hypersurface. If the tangency of $C \in Z_{C_i}^\circ$ at p is simple, then the equation of $C|_{C_i}$ has a double root at p and simple roots elsewhere. By Lemma 2.8, Z_{C_i} is smooth at $[C]$, and its projective tangent space in $\mathbb{P}^5$ is the hyperplane H_p . ■

This identification aligns well with our geometric intuition: a simple tangency of C with C_i at p is equivalent to an order 2 vanishing condition at p . By taking the tangent space, we *linearize* Z_{C_i} at $[C]$ and only remember the first order vanishing condition at p .

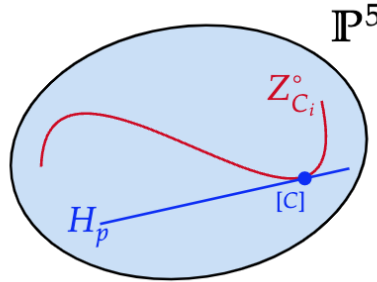


Figure 2.10: $T_{[C]}Z_{C_i}^\circ$ is the linearization H_p .

Corollary 2.11. *Then intersection $Z_{C_1} \cap \cdots \cap Z_{C_5}$ in the space of complete conics is transverse.*

Proof

We first identify $Z_{C_1} \cap \cdots \cap Z_{C_5} \subseteq X$ with $Z_{C_1}^\circ \cap \cdots \cap Z_{C_5}^\circ \subseteq \mathbb{P}^5$. Then in $\mathbb{P}^5$, the (projective) tangent space at each $[C] \in Z_{C_1}^\circ \cap \cdots \cap Z_{C_5}^\circ$ is cut out by the hyperplanes $H_{p_1}, \dots, H_{p_5}$, so it is zero-dimensional. ■

(c) The Chow ring of the space of complete conics. Each Z_{C_i} is a hypersurface in the space of complete conics, so we only need to understand the structure of $A^1(X)$ to determine the class $\zeta = [Z_{C_i}]$. Since $X \subseteq \mathbb{P}^5 \times \mathbb{P}^{5^\vee}$ is a smooth projective subvariety, we may pull back the hyperplane classes from the product of projective spaces. So let $\alpha, \beta \in A^1(X)$ be the pullbacks to $X \subset \mathbb{P}^5 \times \mathbb{P}^{5^\vee}$ of the hyperplane classes on $\mathbb{P}^5$ and $\mathbb{P}^{5^\vee}$. These are respectively represented by the divisors

$$A_p = \{(C, C^\vee) \mid p \in C\}$$

(for any point $p \in \mathbb{P}^2$) and

$$B_L = \{(C, C^\vee) \mid L \in C^\vee\}$$

(for any point $L \in \mathbb{P}^{2^\vee}$).

Also, let $\gamma, \varphi \in A^4(X)$ be the classes of the curves Γ and Φ that are the pullbacks to X of general lines in $\mathbb{P}^5$ and $\mathbb{P}^{5^\vee}$. These are, respectively, the classes of the loci of complete conics $(C, C^\vee)$ such that C contains four general points in the plane, and such that $C^\vee$ contains four points $L_i \in \mathbb{P}^{2^\vee}$ (that is, C is tangent to four lines in $\mathbb{P}^2$). We are now ready to describe the Chow group of divisors on X .

Proposition 2.12. *The group $A^1(X)$ of divisor classes on X has rank 2, and is generated over $\mathbb{Q}$ by α and β . The intersection number of these classes with γ and φ are given by the table*

$$\begin{array}{cc} & \alpha & \beta \\ \gamma & 1 & 2 \\ \varphi & 2 & 1 \end{array}$$

Proof

The intersection numbers are calculated by applying Bezout's theorem. The more interesting part of the statement lies in the Picard rank of X .

We have already found two independent elements α, β of $A^1(X)$, so the Picard number of X is at least 2. To bound the Picard number of X , we let U be the open set of *smooth* conics and consider the following two inclusions

$$\begin{aligned} U &\hookrightarrow \mathbb{P}^5 \hookleftarrow \mathbb{P}^5 \setminus U \\ U &\hookrightarrow X \hookleftarrow X \setminus U. \end{aligned}$$

(Here, we are identifying U as an open set of both $\mathbb{P}^5$ and X). These induce excision sequences

$$\begin{aligned} A^1(\mathbb{P}^5 \setminus U) &\rightarrow A^1(\mathbb{P}^5) \rightarrow A^1(U) \rightarrow 0 \\ A^1(X \setminus U) &\rightarrow A^1(X) \rightarrow A^1(U) \rightarrow 0. \end{aligned}$$

The closed set $\mathbb{P}^5 \setminus U$ is the locus of singular conics, so it is a hypersurface in $\mathbb{P}^5$, say degree d . Then, the first sequence becomes

$$\mathbb{Z} \xrightarrow{\cdot d} \mathbb{Z} \rightarrow A^1(U) \rightarrow 0$$

which implies that the latter is torsion:

$$A^1(U) \cong \mathbb{Z}/d.$$

On the other hand, $X \setminus U$ is the locus of singular *complete* conics, which has two irreducible components $D_1 = \{(L \cup M, 2p^\vee) \mid p \in \mathbb{P}^5\}$ and $D_2 = \{(2L, p^\vee \cup q^\vee) \mid L^\vee \in \mathbb{P}^{5^\vee}\}$, corresponding to two of the three singularity types (recall Section 2.1). Each of these loci is isomorphic a discriminant hypersurface in $\mathbb{P}^5$, so they both have Picard number 1. Thus,

$$\mathrm{rk}_{\mathbb{Z}\mathbf{Mod}} A^1(X \setminus U) = 2.$$

Now, let's supply a kernel term to the second sequence:

$$\begin{array}{ccccccc}
 & 0 & & & 0 & & \\
 & \searrow & & & \nearrow & & \\
 & & \ker f & & & & \\
 & \nearrow & & \searrow & & & \\
 A^1(X \setminus U) & \longrightarrow & A^1(X) & \xrightarrow{f} & A^1(U) \cong \mathbb{Z}/d & \longrightarrow & 0
 \end{array}$$

Since rank is additive, the diagonal short exact sequence implies

$$\mathrm{rk} A^1(X) = \mathrm{rk} \ker f + \mathrm{rk} \mathbb{Z}/d = \mathrm{rk} \ker f.$$

Since $A^1(X \setminus U)$ surjects onto $\ker f$, we have that $\mathrm{rk} \ker f \leq 2$. Hence,

$$\mathrm{rk} A^1(X) \leq 2,$$

so we must have

$$\mathrm{rk} A^1(X) = 2.$$

■

Now, we are ready to calculate the class $\zeta = [Z_{C_i}] \in A(X)$. Since $A^1(X) \otimes \mathbb{Q} \cong \mathbb{Q} \cdot \{\alpha, \beta\}$, we may assume that

$$\zeta = q\alpha + r\beta$$

for $q, r \in \mathbb{Q}$. Recall from our naïve solution that $Z_{C_i}^\circ$ has degree 6 in the set of smooth conics U , so

$$\begin{aligned} \deg \zeta \cdot \gamma &= \gamma(q\alpha + r\beta) \\ &= q + 2r \\ &= 6. \end{aligned}$$

By symmetry,

$$2q + r = 6,$$

so

$$q = r = 2,$$

and

$$\zeta = 2\alpha + 2\beta \in A^1(X).$$

Then, our goal is to find the intersection number

$$\deg \zeta^5 = 32 \deg(\alpha + \beta)^5.$$

So we would like to calculate the intersection products $\alpha^{5-i}\beta^i$, and by symmetry we only have to do so for $i = 0, 1, 2$. Furthermore, since conics passing through 3 general points all have at least rank 2, these intersections will occur only on the open set U_1 where the rank is at least 2, which is isomorphic to the same locus in $\mathbb{P}^5$. Therefore, it suffices to do more Bezout's theorem exercises in $\mathbb{P}^5$.

- $i = 0$: Passing through a point is a linear condition on quadrics. There is a unique quadric through five general points, and the intersection of five hyperplanes in $\mathbb{P}^5$ has degree 1, so $\deg(\alpha^5) = 1$.
- $i = 1$: The quadrics tangent to a given line form a quadric hypersurface in $\mathbb{P}^5$. Since not all conics in the one-dimensional linear space of conics through four general points will be tangent to a general line, $\deg(\alpha^4\beta) = 2$.
- $i = 2$: Similarly, we see that the conics passing through three given general points and tangent to a general line form a conic curve in $U_1 \subset \mathbb{P}^5$. Then, $\deg(\alpha^3\beta^2)$ is the degree of the zero-dimensional intersection of a plane with two quadrics, that is, 4.

Finally, we have everything necessary to solve the 5 conics problem.

Theorem 2.13. *There are 3264 smooth plane conics tangent to five general plane conics.*

Proof

$$\begin{aligned}\deg \left((\alpha + \beta)^5 \right) &= \binom{5}{0} + 2\binom{5}{1} + 4\binom{5}{2} + 4\binom{5}{3} + 2\binom{5}{4} + \binom{5}{5} \\ &= 1 + 10 + 40 + 40 + 10 + 1 \\ &= 102.\end{aligned}$$

Therefore,

$$\begin{aligned}\deg \zeta^5 &= 32 \cdot 105 \\ &= \mathbf{3264!}\end{aligned}$$



3 Solution 2: Excess Intersection Formulas

Modifying naïve parameter spaces is not easy and requires great mathematical insight. Neither is there guarantee that a satisfactory compactification of a parameter space can always be found. Fortunately, excess intersection formulas as developed by Fulton and MacPherson provides means to compute the contribution of excess intersections to our naïve counts via brute force.

Let c, s denote the total Chern and Segre classes. Here is the main theorem:

Theorem 3.1 (Excess intersection formula, Eisenbud and Harris 2016 Theorem 13.3). *If $S \subset X$ is a subvariety of a smooth variety X and T is a locally complete intersection subvariety of X , then*

$$[S][T] := \sum_C (\iota_C)_* (\gamma_C)$$

where:

- The sum is taken over the connected components C of $S \cap T$.
- $\iota_C : C \rightarrow X$ denotes the inclusion morphism.
- $\gamma_C = \left\{ s(C, S) c \left(\mathcal{N}_{T/X}|_C \right) \right\}_d \in A_d(C)$, where $d = \dim X - \operatorname{codim} S - \operatorname{codim} T$ is the "expected dimension" of the intersection.

Furthermore, if the subvariety S is locally a complete intersection as well, then we have a symmetric form

$$\gamma_C = \left\{ s(C, X) c \left(\mathcal{N}_{S/X}|_C \right) c \left(\mathcal{N}_{T/X}|_C \right) \right\}_d.$$

This formula is of crucial importance in the development of intersection theory. First, it provides a satisfactory generalization of intersection products on *singular* varieties. Second, it suffices to define the product on local complete intersections and use the formula to extend to arbitrary subvarieties. Third, the formula provides a way to define general pullbacks without assuming smoothness or the moving lemma. See W. Fulton [1984](#) §6 for a detailed treatment (the excess intersection formula is stated in §6.3).

3.1 Solution 2 to the Five Conics Problem

Recall,

$$\bigcap_{i=1}^5 Z_i = T \cup \Gamma,$$

where T denotes the excess intersection (supported on the Veronese surface S of double lines) and Γ the dimension-zero locus corresponding to smooth tangent conics. By the excess intersection formula applied to $\mathbb{P}^5$,

$$\begin{aligned} 7776 &= [Z_1] \cdots [Z_5] \\ &= (\iota_S)_*(\gamma_S) + \sum_{p \in \Gamma} (\iota_p)_*(\gamma_p) \\ &= \{s(T, \mathbb{P}^5) \prod_i c(\mathcal{N}_{Z_i/\mathbb{P}^5}|_S)\}_0 + \deg \Gamma \\ &= \deg \left(s(T, \mathbb{P}^5) \prod_i c(\mathcal{N}_{Z_i/\mathbb{P}^5}|_S) \right) + \deg \Gamma. \end{aligned}$$

Thus, the answer to the five conics problem will be given by

$$\deg \Gamma = 7776 - \deg \left(s(T, \mathbb{P}^5) \prod_i c(\mathcal{N}_{Z_i/\mathbb{P}^5}|_S) \right).$$

To complete this, we need to calculate

- (a) $c(\mathcal{N}_{Z_i/\mathbb{P}^5}|_S)$ for each i , and
- (b) $s(T, \mathbb{P}^5)$.

(a) Let $\zeta \in A^1(S)$ be the class of a line in $S \cong \mathbb{P}^2$, and let $\eta \in A^1(\mathbb{P}^5)$ be the hyperplane class of $\mathbb{P}^5$. Since S is a quadric surface, we see that

$$\eta|_S = 2\zeta.$$

We have seen in our naïve computation that Z_i are sextic hypersurfaces, so

$$\mathcal{N}_{Z_i/\mathbb{P}^5} \cong \mathcal{O}_{Z_i}(6),$$

which implies that

$$c(\mathcal{N}_{Z_i/\mathbb{P}^5}) = 1 + 6\eta|_S = 1 + 12\zeta.$$

(b) Hiding some details under the rug, we first compute

$$s(S, X) := s(\mathcal{N}_{S/\mathbb{P}^\vee}) = \frac{1}{c(\mathcal{N}_{S/\mathbb{P}^5})}.$$

By the normal bundle sequence

$$0 \rightarrow \mathcal{T}_S \rightarrow \mathcal{T}_{\mathbb{P}^5}|_S \rightarrow \mathcal{N}_{S/\mathbb{P}^5} \rightarrow 0,$$

we have

$$c(\mathcal{N}_{S/\mathbb{P}^5}) = \frac{c(\mathcal{T}_{\mathbb{P}^\nabla}|_S)}{c(\mathcal{T}_S)}.$$

Recall that the total Chern class of $\mathcal{T}_{\mathbb{P}^n}$ is $(1 + [H])^{n+1}$. (This can be deduced by the Euler sequence). Since $S \cong \mathbb{P}^2$, we have

$$c(\mathcal{T}_S) = (1 + \zeta)^3 = 1 + 3\zeta + 3\zeta^2.$$

Similarly,

$$\begin{aligned} c(\mathcal{T}_{\mathbb{P}^5}|_S) &= (1 + \eta)^6|_S \\ &= 1 + 6\eta_S + 15\eta_S^2 \\ &= 1 + 12\zeta + 60\zeta^2. \end{aligned}$$

Thus,

$$c(\mathcal{N}_{S/\mathbb{P}^5}) = \frac{1 + 12\zeta + 60\zeta^2}{1 + 3\zeta + 3\zeta^2} = 1 + 9\zeta + 30\zeta^2,$$

whence

$$s(S, X) = \frac{1}{1 + 9\zeta + 30\zeta^2} = 1 - 9\zeta + 51\zeta^2.$$

Now, the component of $\bigcap Z_i$ supported on S is exactly the scheme $T = V(\mathcal{I}_{S/\mathbb{P}^5}^2)$ defined by the square of the ideal $\mathcal{I}_{S/\mathbb{P}^5}$. This implies that the degree- k term in $s(T, \mathbb{P}^5)$ is exactly 2^{k+3} times the term in $s(S, \mathbb{P}^5)$. Hence,

$$s(T, X) = 8 - 144\zeta + 1632\zeta^2.$$

Now, we are ready to calculate the contribution to 7776 by the excess intersection:

$$\deg \left(s(T, \mathbb{P}^5) \prod_i c(\mathcal{N}_{Z_i/\mathbb{P}^5}|_S) \right) = \deg(1 + 12\zeta)^5 (8 - 144\zeta + 1632\zeta^2)$$

which is the coefficient in front of ζ^2 , and that is

$$1632 - 144 \cdot 5 \cdot 12 + 8 \cdot 10 \cdot 12^2 = 4512.$$

Therefore, we see again that

$$\begin{aligned} \deg \Gamma &= 7776 - 4512 \\ &= 3264! \end{aligned}$$

4 Miscellaneous

Like 27 lines on a cubic, the literature on 3264 conics is quite extensive. Below are a few extra sources that might be of interest to the reader.

4.1 Solution 3: Equivariant Cohomology

Anderson and William Fulton [2023](#) §6 details a computation of $\deg(2\alpha + 2\beta)^5$ as an equivariant integral. The reader may find the weight diagrams of the torus action on the space of complete conics quite interesting.

4.2 Sottile's 3264 Real Conics

Sottile [2008](#) outlines an argument that all 3264 tangent conics can in fact be real conics. Here is an example:

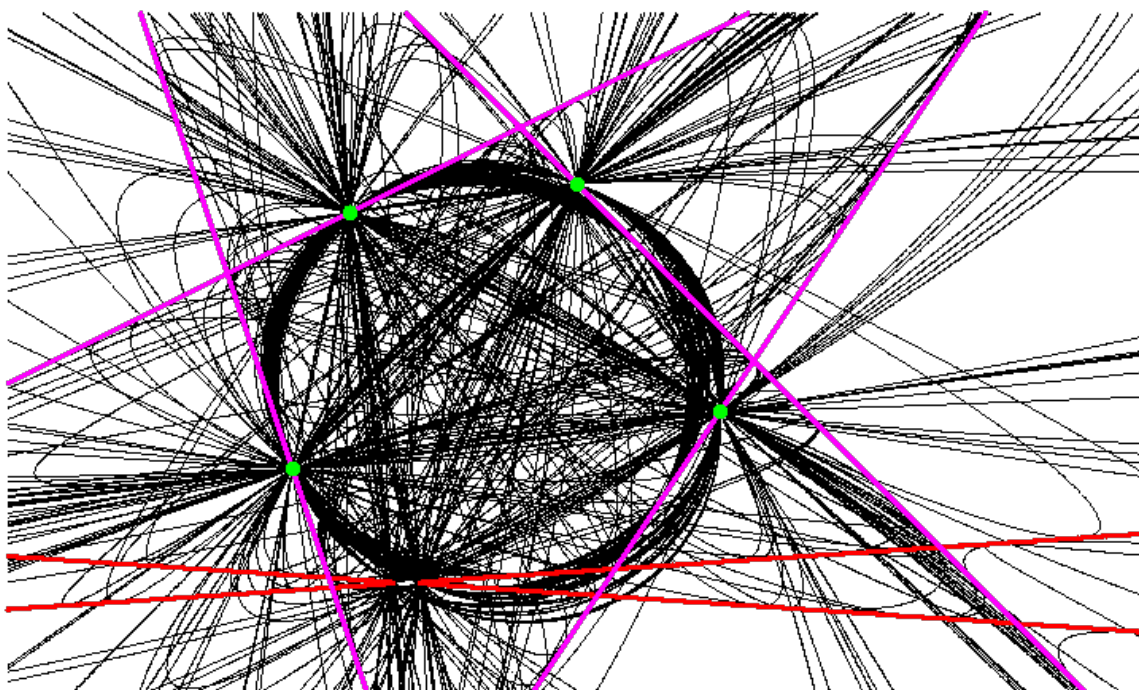


Figure 4.1: Sottile's 3264 real conics.

4.3 Visualizing All 3264 Conics

S. T. Paul Breiding B. S. 2020 applies techniques in numerical algebraic geometry to quickly generate all 3264 (real or complex) tangent conics given 5 arbitrarily chosen conics. See <https://www.juliahomotopycontinuation.org/examples/3264/>. They provide a do-it-yourself page where the user may randomly choose their favorite 5 conics, and then the program outputs a demonstration of the 3264 tangent conics in an instant. Sadly, the website does not seem to be working at the moment.

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