Plücker's Formulae

&

Monodromy of Flexes

Outline

- Plücker formulae

  for flexes & bitangents

- Monodromy of flexes

  general case $\quad S_{3d(d-2)}$

- Monodromy of flexes

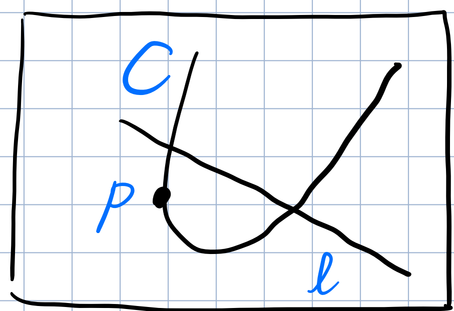
  plane cubics $\quad ASL_2(\mathbb{F}_3)$

Dual Projective Plane

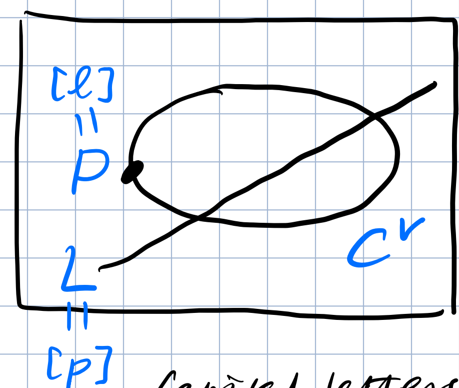
Convention:

\vee

$\mathbb{P}^2$ $\qquad$ $\mathbb{P}^{2\vee}$



Lower case letters



Capital letters

$$p = (x:y:z)$$

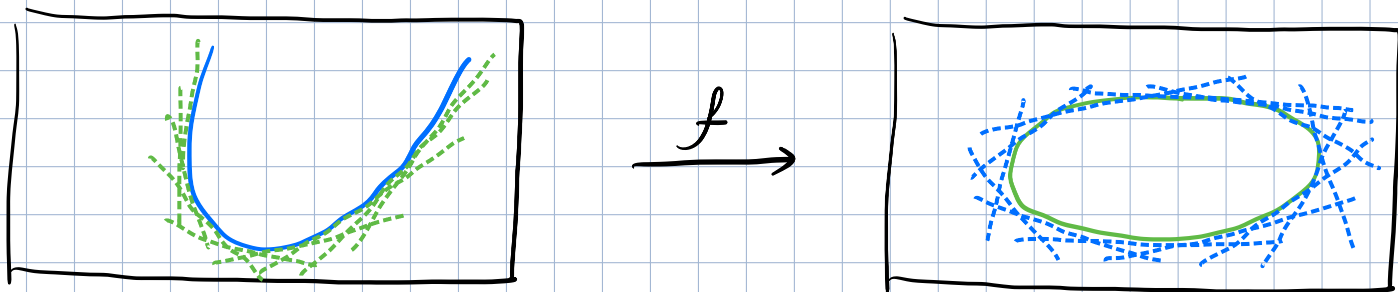
$$P = [l] = [Z(ax+by+cz)]$$
$$= (a:b:c)$$

Dual Plane Curve

$C = Z[F] \subseteq \mathbb{P}^2$ smooth

$$\Rightarrow f : C \to C^\vee$$

$$p \longmapsto [T_pC]$$



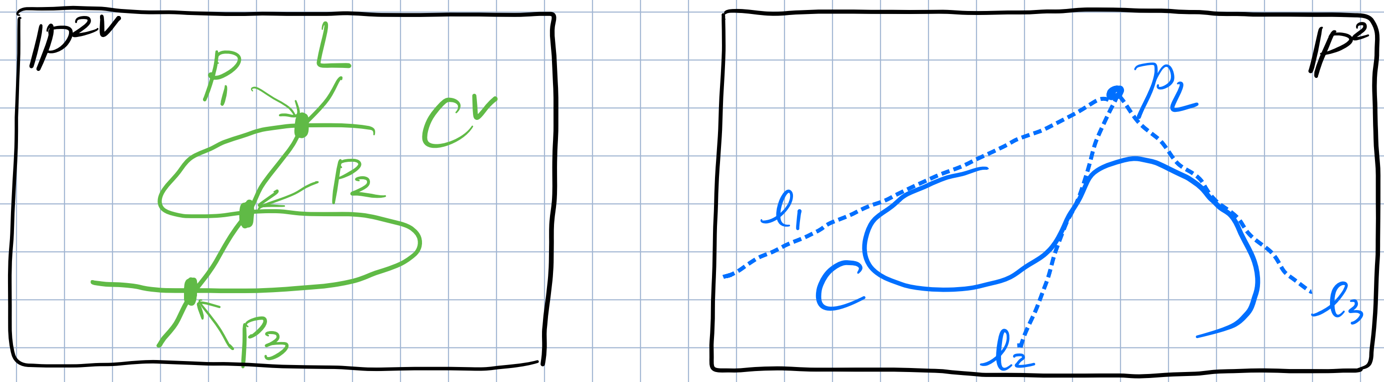
$$p \longmapsto [T_pC] = [xF_x(p) + yF_y(p) + zF_z(p) = 0]$$

$$= (F_x(p) : F_y(p) : F_z(p)).$$

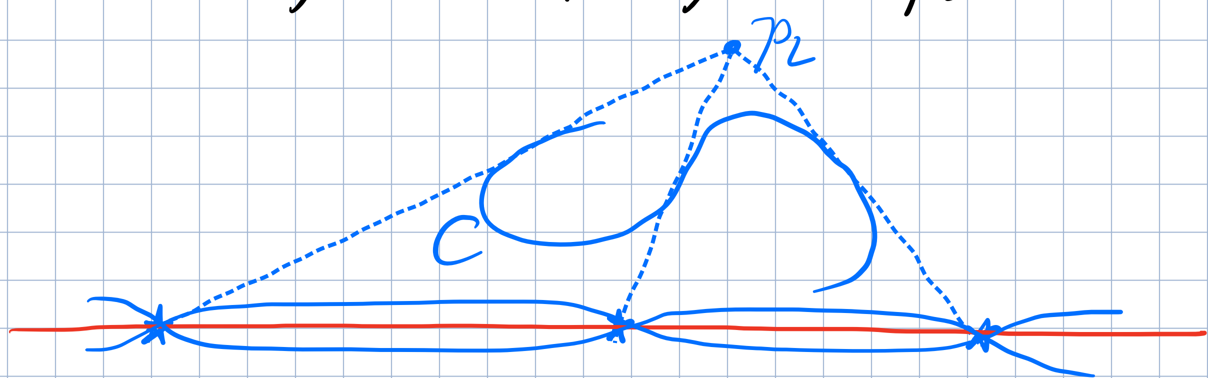
Cheerful fact: $(C^\vee)^\vee = C$.

Dual Plane Curve : Degree

$$\deg C^\vee = \deg(C^\vee \cdot L) = \#\text{points of } C^\vee \cap \text{generic } L$$



$= \#$ tangent lines thru general $p_L \notin C$



$= \#$ ramification pts of $\pi_{p_L} : C \to \mathbb{P}^1$.

Dual Plane Curve: Degree

$\deg C = d \quad \Rightarrow \quad \pi_{pL}$ is generally $d:1$

Riemann-Hurwitz: $\quad C_1 \xrightarrow{\pi} C_2$

$$\rightsquigarrow \quad 2g_1 - 2 = d \cdot (2g_2 - 2) + r$$

$$\rightsquigarrow \quad 2\binom{d-1}{2} - 2 = d \cdot (0-2) + r$$

$$\rightsquigarrow \quad r = d(d-1)$$

$$\boxed{\deg C^\vee =: d^\vee = d(d-1)}$$

Dual Curve: Genus

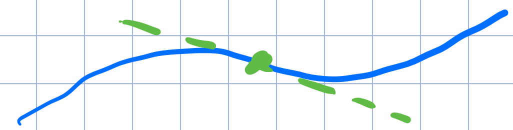
C, $C^\vee$ birational

$\Rightarrow$ same geometric genus

(= genus of normalization)

Flexes & Bitangents

For a smooth plane curve C, a flex is a pair (p, ℓ)
s.t. $m_p(C \cdot \ell) \geq 3$



A bitangent is a triple (p_1, p_2, ℓ) s.t. $m_{p_1}(C \cdot \ell), m_{p_2}(C \cdot \ell) \geq 2$

A general smooth plane curve has only ordinary $(m_p = 3)$ flexes & bitangents.

Flexes & Bitangents

$\boxed{\text{Flexes}}$ Say $mp(C\cdot \ell) = m$

Then locally parametrically, $C \rightsquigarrow v(t) = (1 : t : t^m + \text{h.o.t.})$

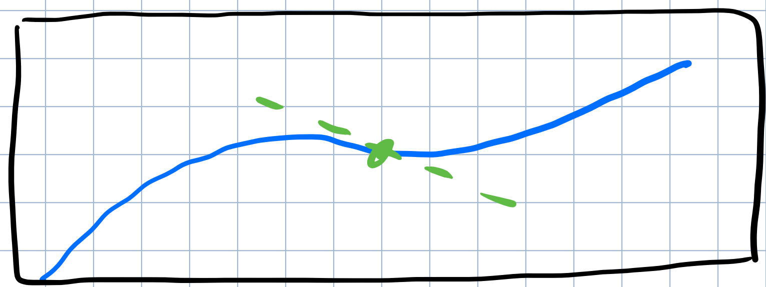
Dual coords $(a:b:c)$ $\begin{cases} \text{passes thru } v(t) & (a,b,c)\cdot v(t) = 0 \\ \text{tangent to } v(t) & (a,b,c)\cdot v'(t) = 0 \end{cases}$

$$\Rightarrow (a,b,c) = \vec{v}(t) \times \vec{v}\,'(t)$$

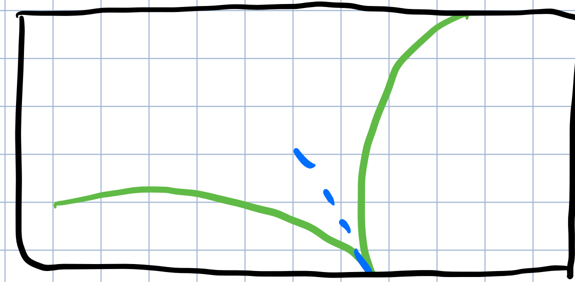
$$= (1 + \cdots, \; m t^{m-1} + \cdots, \; (m-1) t^m + \cdots)$$

cusp of ord. m

Flexes & Bitangents

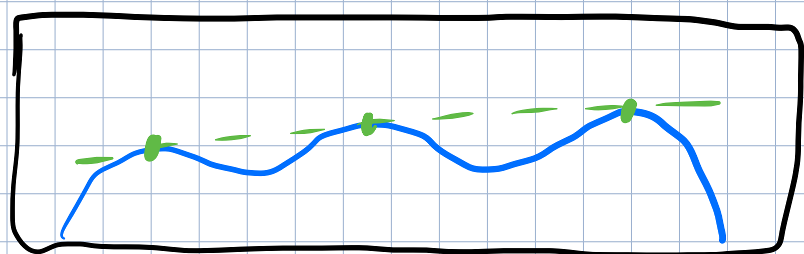


Flex of ord. m

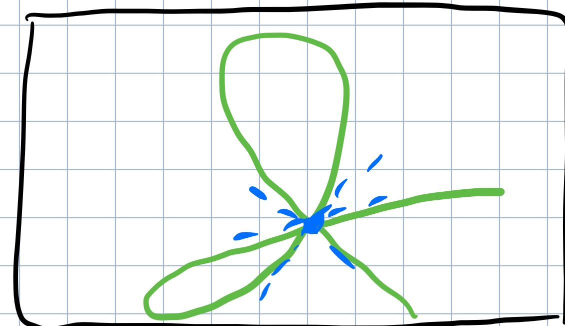


Cusp of ord. m

Bitangents



Bitangent of ord. m



node of ord m

Plücker Formulae

Say $C\cdot\ell = \sum_\alpha m_\alpha\cdot p_\alpha$

Def $m(\ell) = m_C(\ell) := \sum_\alpha (m_\alpha - 1)$ extra degree of tangency

$e(\ell) = e_C(\ell) := \#\alpha$. extra points of tangency

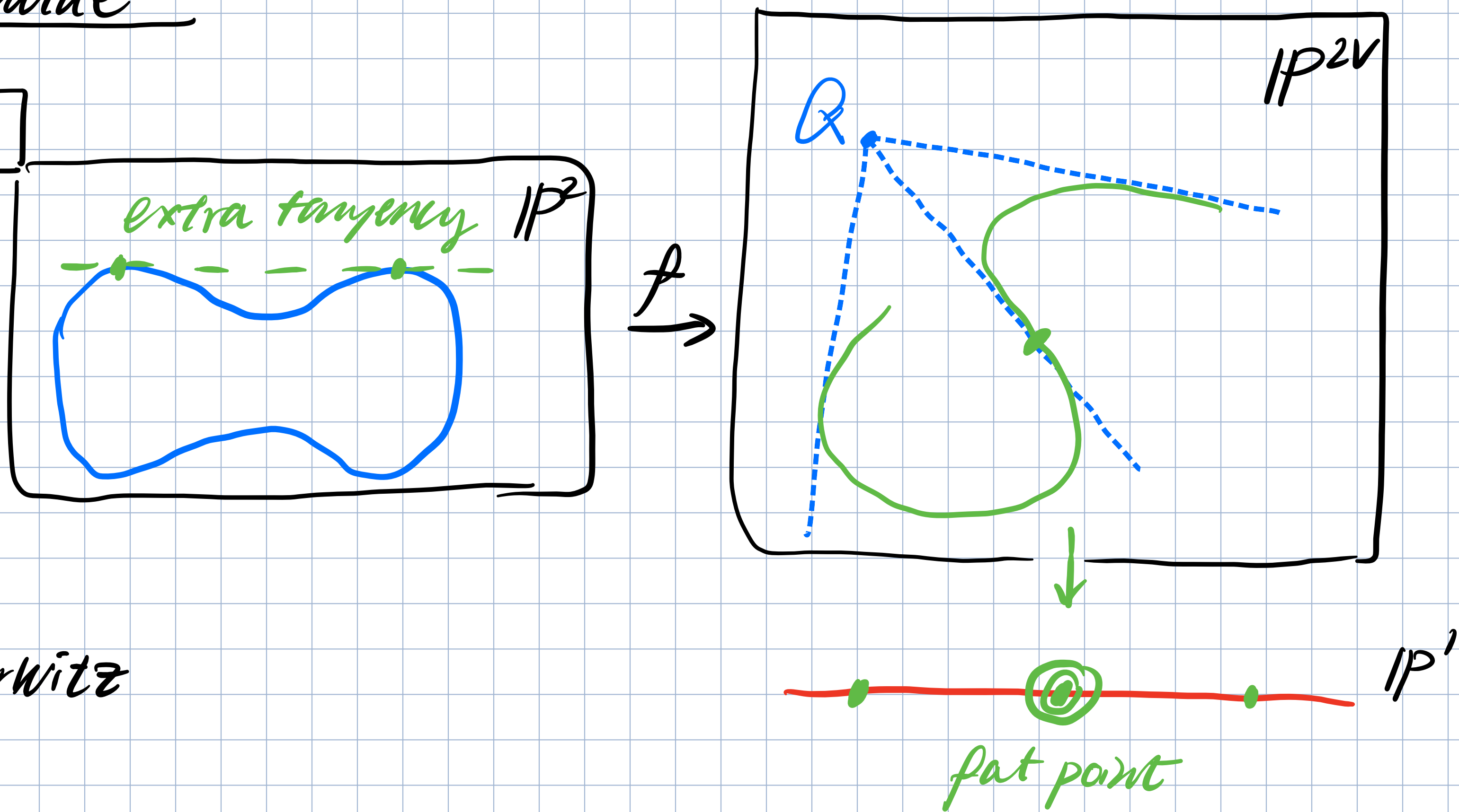
$\boxed{\text{Plücker I}}$ $g_C = g_{C^\vee}$

$\rightsquigarrow$ $$\binom{d-1}{2} = \binom{d^\vee - 1}{2} - \sum_{[\ell]\in\mathbb{P}^{2\vee}} \binom{m(\ell)}{2}$$

$\rightsquigarrow$ $$\sum_\ell \binom{m(\ell)}{2} = \binom{d(d-1)-1}{2} - \binom{d-1}{2}$$

Plücker Formulae

$\boxed{\text{Plücker II}}$



Riemann-Hurwitz

$$\rightsquigarrow \quad r = d + \sum_{\ell} (m(\ell) - e(\ell))$$
$$= d(d-3) + 2d(d-1)$$

$$\rightsquigarrow \quad \sum_{\ell} (m(\ell) - e(\ell)) = 3d(d-2)$$

Plücker Formulas: Flexes & Bitangents

- Ordinary flexes: $m(\ell)=2$, $e(\ell)=1$
- Bitangents: $m(\ell)=2$, $e(\ell)=2$

f: # flexes b: # bitangents

$$\rightsquigarrow \quad f = 3d(d-2)$$

$$b = \frac{1}{2}d(d-2)(d-3)(d+3)$$

Eg/ $d=3$. $f=9$, $b=0$ Eg/ $d=4$ $f=24$ $b=28$.

Flex Divisor & Hessian Curve

$C = Z[F] \subseteq \mathbb{P}^2$ smooth plane curve

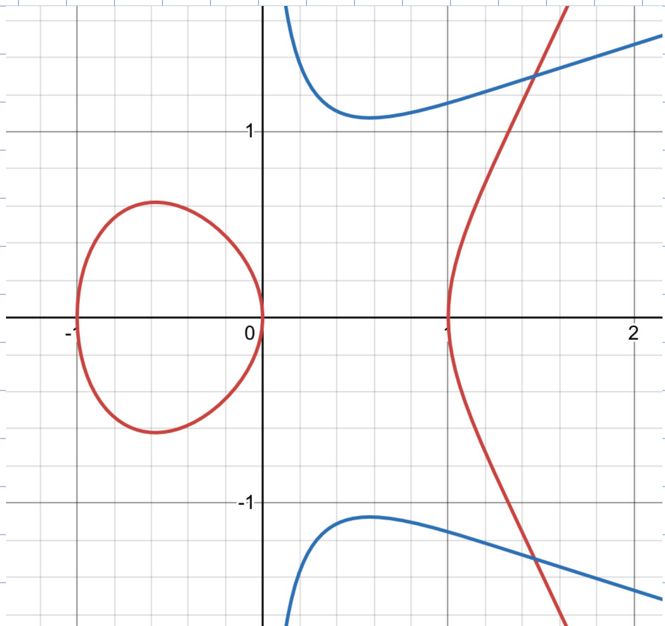
$$\mathrm{Hess}(C) := Z[\det HF]$$
$$= Z[\det(\partial_{ij}F)_{i,j}]$$

(degree $3(d-2)$)

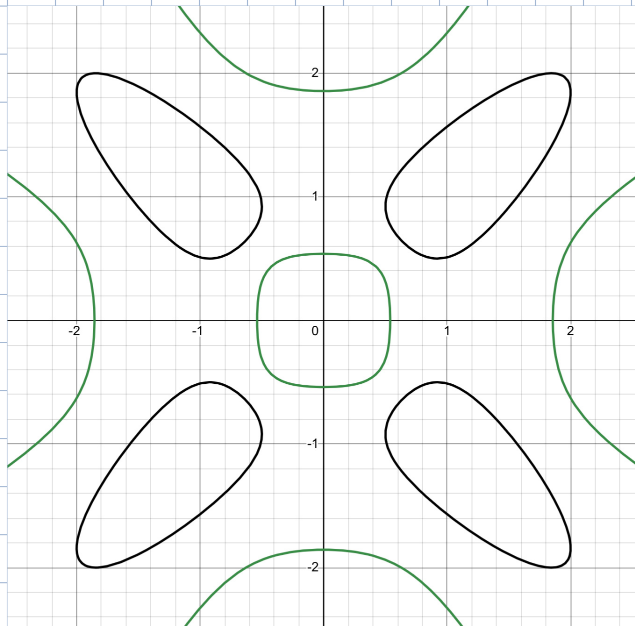
<u>Cheerful fact</u> [EH25, Theorem 13.6]

$$\mathrm{Flex}(C) = C \cdot \mathrm{Hess}(C)$$

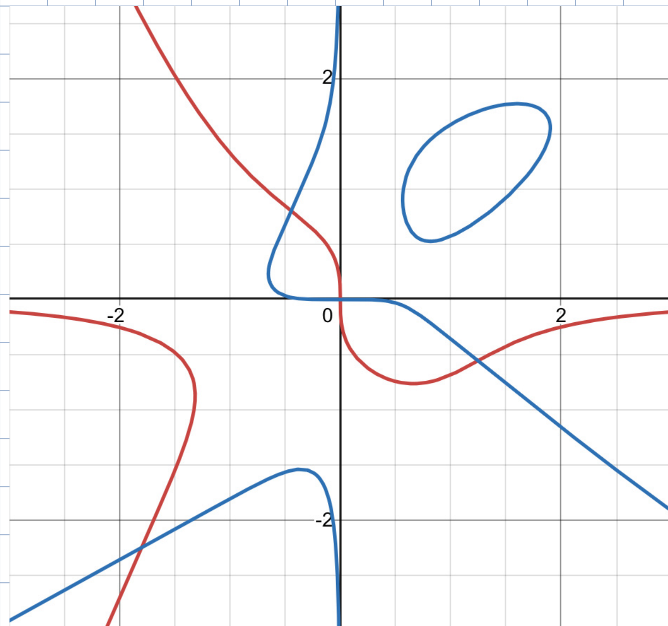
Flex Divisor & Hessian Curve



Weierstraß cubic



Edge quartic



Klein quartic

Incidence Correspondence of Flexes

$W_d := \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^2}(d)))$ compl. lin. system of deg d plane curves.

$I_0 := \{(p,[\ell]) \mid p \in \ell\} \subseteq \mathbb{P}^2_{x,y,z} \times \mathbb{P}^{2\vee}_{a,b,c}$ $\quad ax+by+cz=0$

$I_d := \{(C,p,\ell) \mid m_p(C\cdot\ell) \geq 3\} \subseteq W_d \times I_0$

Incidence correspondence of flexes of deg d plane curves

$$\begin{tikzcd} & I_d \arrow[dl, "\pi"', "3d(d-2):1"] \arrow[dr, "\eta"] & \\ W_d & & I_0 \end{tikzcd}$$

(C,p,ℓ) $\quad C \quad (p,\ell)$

Q: Monodromy of this cover?

Monodromy of Flexes: General Case

Characterization of S_n

S_n = full symmetric group

2-transitive: transitive on pairs

$$(i, j) \xmapsto{\exists g} (k, l)$$

Transposition: swaps exactly 2 elements.

$$i \longleftrightarrow j$$

Lem G finite, 2-transitive, contains transposition

$\Rightarrow$ $G \cong S_n$

(proof: has all transpositions).

Monodromy of Flexes: Irreducibility

Let M be the monodromy group of $\begin{array}{c} I_d \\ \downarrow \\ W_d \end{array}$.

Need: I_d irreducible.

Recall:

$$\begin{tikzcd} \underset{(C,p,\ell)}{} I_d \arrow[r] \arrow[d] & \underset{(p,\ell)}{} I_0 \\ \underset{C}{} W_d & \end{tikzcd}$$

fiber $= \{C \mid m_p(C\cdot\ell) \geq 3\}$

3 linear conds

$\Rightarrow$ codim 3 lin. subspace of W_d

In particular: irreducible

$\rightsquigarrow$ I_d is proper / W_d irreducible, with irred. equidimensional fibers.

Monodromy of flexes: Irreducibility

Cheerful fact $\underset{\text{var}}{X} \longrightarrow \underset{\text{irred. var}}{Y}$ + irred. equidimensional fibers.

Then either – X proper / Y

or – X pure dimensional

$\Rightarrow$ X irreducible

$\rightsquigarrow$ Id irreducible.

connected $\Longleftarrow$ irreducible

General theory:

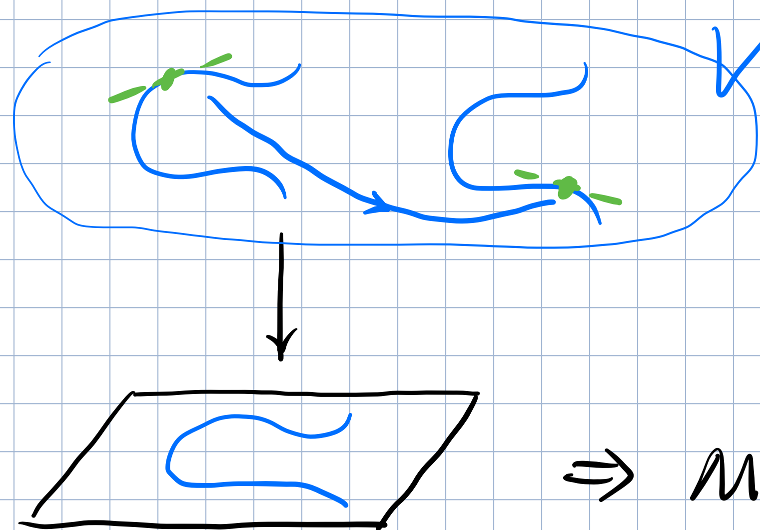
$$\begin{tikzcd} V \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Id \arrow[d] \\ U \arrow[r] & Wd \end{tikzcd}$$

unramified locus

Monodromy of flexes: 2-transitive

V connected

$$\begin{array}{c} V \\ \downarrow \\ U \end{array}$$



$\Rightarrow M$ transitive.

2-transitive $\iff$ $\mathrm{Stab}(p_0, \ell_0)$ transitive on other (p', ℓ').

Fix (C_0, p_0, ℓ_0). $I' := \{(C, p, \ell) \mid m_p(C \cdot \ell) \geq 3,\ (p, \ell) \neq (p_0, \ell_0)\}$ irred

$$\begin{array}{c} I' \\ \downarrow \\ W' = \{C \mid m_{p_0}(C \cdot \ell_0) \geq 3\} \subseteq W_d \end{array}$$

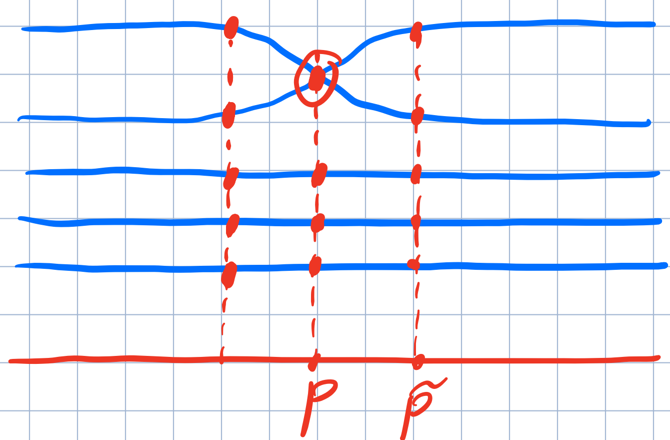
$\Rightarrow V \cap I'$ connected $\Rightarrow$ $\mathrm{Stab}(p_0, \ell_0)$ transitive on others

Monodromy of Flexes : Transposition

Key lemma $\pi : Y \to X$ holomorphic, degree n.

If there is $p \in X$ s.t. the fiber Y_p breaks up as $n-2$ single pts $q_1, q_2, \dots, q_{n-2}$ and 1 fat point $2 \cdot q_n$

then M contains a transposition.



Pf Take small disk D around p

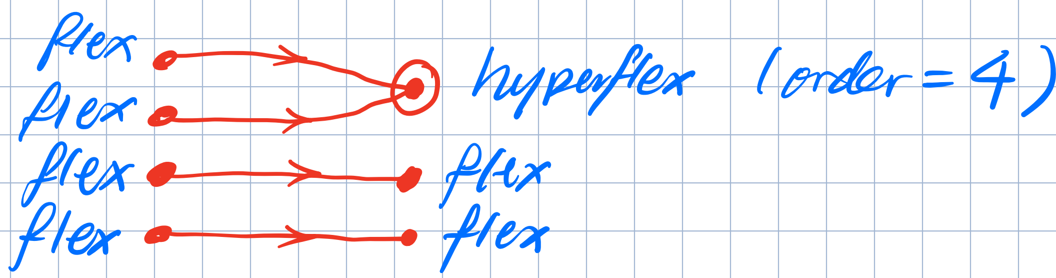
$D \cap U \to$ unramified,

$\exists \tilde{p}$, $Y_{\tilde{p}} = n$ distinct pts $\tilde{q}_1, \tilde{q}_2, \dots, \tilde{q}_n$

The arc $\tilde{q}_{n-1} \to \tilde{q}_n \Rightarrow$ transposition $(n-1, n)$

Monodromy of Flexes: Transpositions

$d \geq 4$



If $d = 3$, then can't have local equation $(1 : t^3 : t^4 + h.o.t.)$

$\rightsquigarrow$ argument breaks down!

$$I'' := \{(C, p, \ell) \mid m_p(C \cdot \ell) \geq 3,\ p \neq p_0,\ \ell \neq \ell_0\}$$

$\downarrow$ Plücker formulae $\Rightarrow$ $(3d(d-2)-1) : 1$

$$W'' := \{C \mid m_{p_0}(\ell_0 \cdot C) \geq 4\}.$$

$\Rightarrow$ general $C \in W''$ has $3d(d-2)-1$ distinct flexes

$\Rightarrow$ contains one hyperflex.

Monodromy of flexes, general case

Thm $d \geq 4$. $M \cong S_{3d(d-2)}$

Monodromy of Flexes:

Plane Cubic

The Jacobian of a Curve

C smooth curve, genus g.

$$H_1(C,\mathbb{Z}) \cong \mathbb{Z}^{2g}, \text{ and } H^0(\omega_C)^\vee \underset{\text{duality}}{\overset{\text{Serre}}{\cong}} H^1(\mathcal{O}_C) \cong \mathbb{C}^g.$$

Have inclusion $i: H_1(C,\mathbb{Z}) \hookrightarrow H^0(\omega_C)^\vee$

path $p \to q \longmapsto$ functional $\int_p^q$ on diff. forms.

Def The Jacobian of C is

$$J(C) := \mathbb{C}^g / i(\mathbb{Z}^{2g}).$$

Choose base point $p_0 \in C$

$\rightsquigarrow$ Abel-Jacobi map $\mu_d : C^d \to J(C)$

$$q \longmapsto \sum_1^d \int_{p_0}^{q_i}$$

Abel's Theorem & Consequences

Abel's Thm $\quad D_1 \sim D_2 \iff \mu_d(D_1) = \mu_d(D_2)$

$$C^d \longrightarrow \mathrm{Pic}_d(C) \xrightarrow{-d\cdot p} \mathrm{Pic}_0(C) \xrightarrow{\cong} J(C)$$

Consequence 1 $\quad$ # of n-torsion elements in $\mathrm{Pic}(C)$ is n^{2g}

Pf $\quad J(C) \cong (S^1)^{2g}$, n-torsions are the subgrp $(\mathbb{Z}/n)^{2g}$

Consequence 2 $\quad E$ elliptic curve.

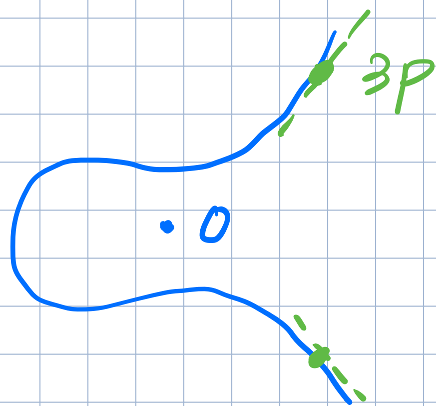
$$E \cong J(E).$$

Flexes of a Plane Cubic

$(E, 0)$ elliptic curve. Complete linear system $|3 0|$

$$\rightsquigarrow \quad \mathbb{C}/\Lambda \xrightarrow[\varphi]{(1:\wp:\wp')} \mathbb{P}^2, \text{ plane cubic } C$$

Hyperplane class $= [3 0]$



(p, ℓ) flex $\Rightarrow \quad 3p \sim 3 0$

$\Rightarrow \quad p - 0$ is 3-torsion in $J(C)$

If $p = \varphi(z)$, then $3z \equiv 3 \cdot 0 \pmod{\Lambda}$

Flexes of a Plane Cubic

$$3z \equiv 3 \cdot 0 \pmod{\Lambda} \implies z - 0 \in \tfrac{1}{3}\Lambda$$

$\rightsquigarrow$ Flexes of plane cubic $= \operatorname{coker} \Lambda \xrightarrow{\cdot 3} \Lambda$

$$\cong (\mathbb{Z}/3) \times (\mathbb{Z}/3)$$

$$\cong \mathbb{A}^2_{\mathbb{F}_3}.$$

Furthermore, recall $\Lambda = H^1(E, \mathbb{Z})$.

intersection form $\Lambda \times \Lambda \to \mathbb{Z}$

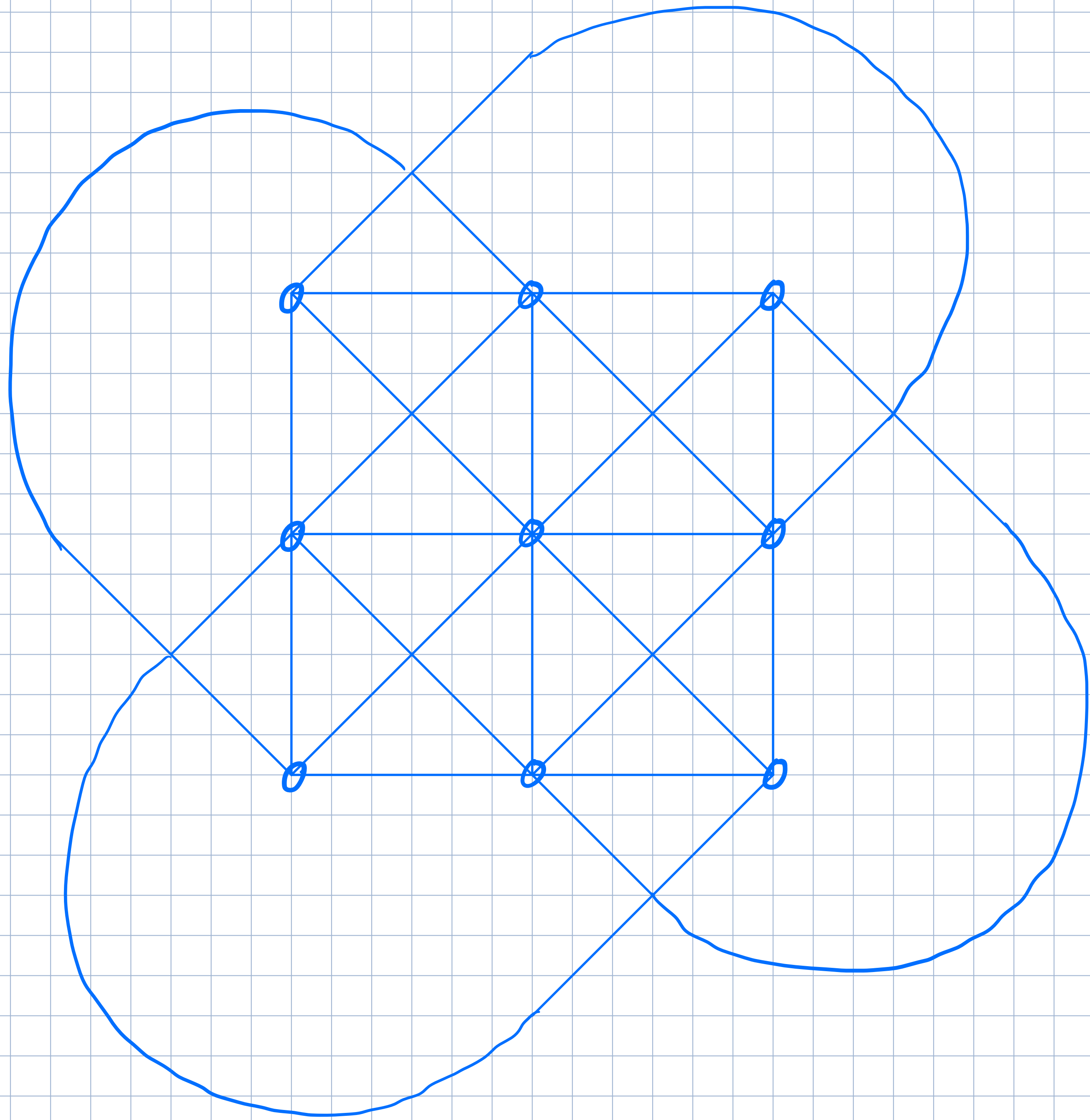
descends to bilinear form

$$\mathbb{A}^2_{\mathbb{F}_3} \times \mathbb{A}^2_{\mathbb{F}_3} \to \mathbb{F}_3$$

Key: Monodromy needs to preserve all this structure!

Flexes of a Plane Cubic

$\mathbb{A}^2_{\mathbb{F}_3}$



Flexes of Plane Cubic: Monodromy

$\mathbb{F}_3^2 \rtimes SL_2(\mathbb{F}_3)$

A priori, M must be a subgroup of $ASL_2(\mathbb{F}_3)$

Prop^n $M \cong ASL_2(\mathbb{F}_3)$

Slogan: (Sottile) Monodromy groups are as large as they can be.

Pf Fix $p \in Flex(C)$.

$$\boxed{\text{I.} \quad Stab(p) \cong SL_2(\mathbb{F}_3)}$$

For each $A \in SL_2(\mathbb{F}_3)$, we construct an arc in $V \subseteq I_3$ fixing (C, p).

Flexes of a Plane Cubic : Monodromy

$\forall A \in SL_2(\mathbb{F}_3)$, pick $\bar{A} \in SL_2(\mathbb{Z})$ s.t.

A = reduction of $\bar{A}$ mod 3.

Recall $E \cong \mathbb{C}/\Lambda$. $\bar{A} \in SL_2(\mathbb{Z})$ fixes Λ

$\rightsquigarrow \mathbb{C}/\bar{A}\cdot\Lambda = \mathbb{C}/\Lambda = E \implies$ same plane cubic C.

Now consider $\bar{A} \in SL_2(\mathbb{R})$.

Draw an arc $A(t)$ in $SL_2(\mathbb{R})$

$$\text{s.t.} \begin{cases} A(0) = I, \\ A(1) = \bar{A}. \end{cases}$$

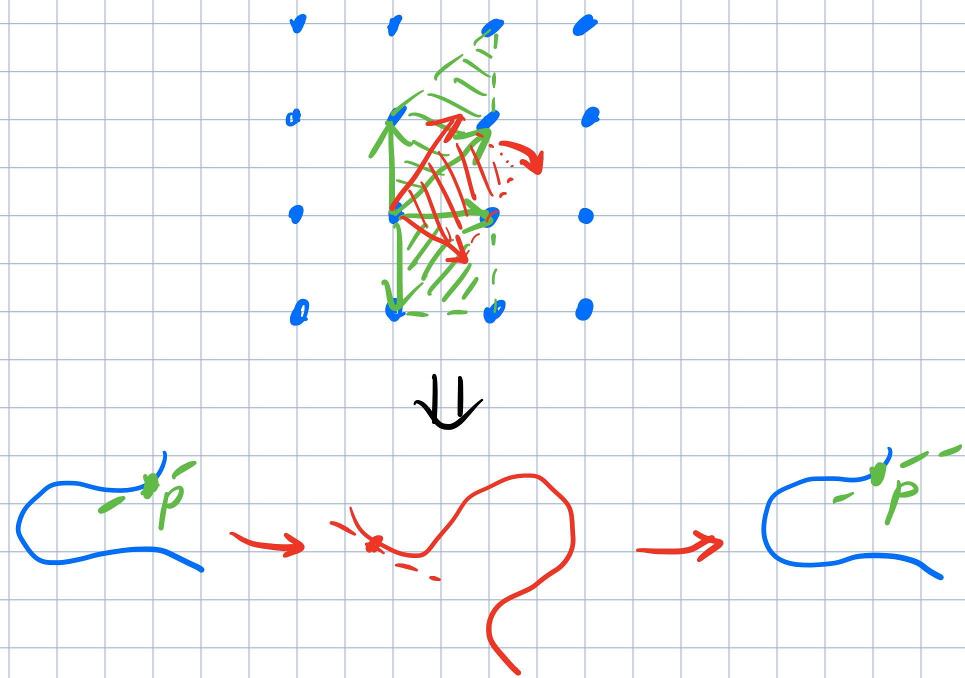
Flexes of a Plane Cubic: Monodromy

$A(t)$: $\quad A(0) = I, \quad A(1) = \bar{A}$.

Consider the 1-parameter family of complex tori

$$E(t) := \mathbb{C} \big/ A(t)\cdot\Lambda$$

I_3



$\rightsquigarrow$ Associated monodromy fixes p & permutes other 8 flexes.

Flexes of a Plane Cubic: Monodromy

$\boxed{\text{II. Assuming I, any } a \in ASL_2(\mathbb{F}_3) \text{ lies in } M}$

Notice that $\mathrm{Stab}_{ASL_2(\mathbb{F}_3)}(p) = \mathrm{Stab}_M(p) \cong SL_2(\mathbb{F}_3)$

$\rightsquigarrow$ Any elt of $ASL_2(\mathbb{F}_3)$ fixing p must be in M.

Take any $a \in ASL_2(\mathbb{F}_3)$.

M is transitive $\Rightarrow \exists m \in M,\ m \cdot p = a \cdot p$

$\rightsquigarrow a^{-1}m \cdot p = p$ so lies in M

$$\Rightarrow a \in M$$

$\square$

Formula for Flexes of Plane Cubic ?

$ASL_2(\mathbb{F}_3)$ is solvable

$\rightsquigarrow$ technically possible to give formulae for the 9 flexes using radicals

Q: Has this been written down ?

If not, what's a good way to do it ?

cf. Har79 p. 696 .

Other Monodromy groups

[EH25 Chapter 11] Monodromy of hyperplane sections

Theorem 11.3 Monodromy of hyperplane sections of an irred. curve is S_d.

Severi variety $V_{d,g} := \{ C \overset{\text{irred}}{\subseteq} \mathbb{P}^2 \mid Sing(C) = \binom{d-1}{2} - g \text{ ord. nodes} \}$

$$\begin{array}{l} \Phi := \{ (C, p) \mid p \in Sing(C) \} \\ \downarrow \\ V_{d,g} \end{array}$$

Proposition 11.15 Monodromy of Φ is $S_{\binom{d-1}{2}-g}$.

References

[Har79] Galois groups of enumerative problems

[Zha19] (Minor thesis) Galois groups of enumerative problems

[EH25] The Practice of Algebraic Curves
a second course in algebraic geometry.

Mircea Mustata. "An irreducibility criterion"