Galois Group of Circles of Apollonius

3264 and All That §2.3

What is a circle?

- affine equation $(x-a)^2+(y-b)^2=r^2$
- homogeneous eq. $(x-az)^2+(y-bz)^2=r^2z^2$

$\iff$ passes through circular points

$o_+ = (1:i:0)$, $\quad o_- = (1:-i:0) \quad \in \mathbb{P}^2$.

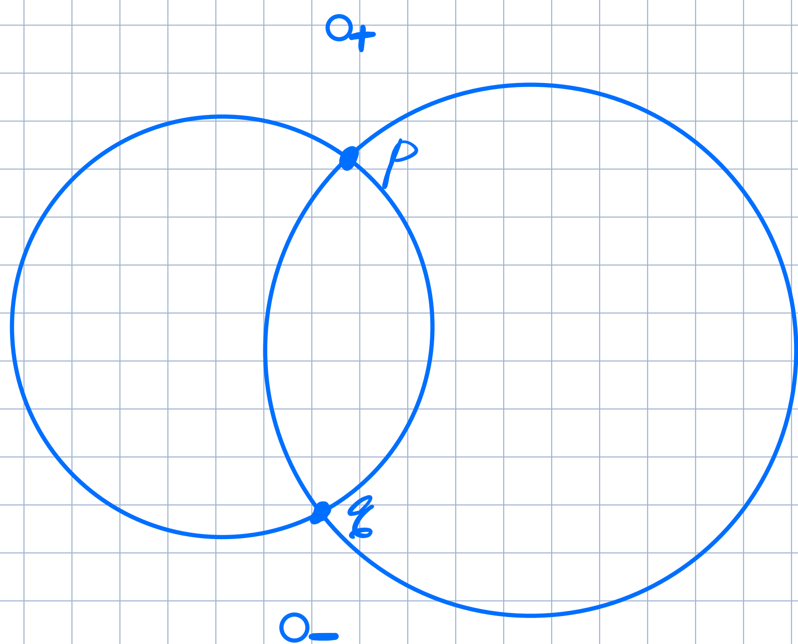
$\iff$ $C = Z[f]$, $f \in$ ideal (z, x^2+y^2)
and $\deg f = 2$.

Parameter space of circles

Passing through o_+, o_- $\rightsquigarrow$ 2 linear conditions on $|H^0(\mathcal{O}_{\mathbb{P}^2}(2))| = \mathbb{P}^5$

So parameter space of circles $\mathcal{C} \cong \mathbb{P}^3 \subseteq \mathbb{P}^5$.

Any 2 circles meet at o_+, o_-, and 2 other points. p, q.



Circles tangent to a given circle

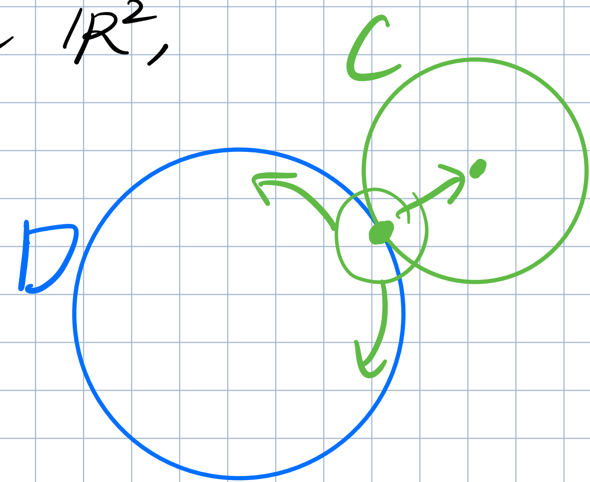
Def Two circles C and D are tangent if $p = q$.

Def Fix a circle $D \in \mathcal{C}$.

Let $Z_D := \{ C \in \mathcal{C} \text{ circle} \mid C \text{ tangent to } D \}$

Prop^n $Z_D \subseteq \mathcal{C} \cong \mathbb{P}^3$ is a quadric hypersurface.

Heuristic argument: in $\mathbb{R}^2$,



2 degrees of freedom.

Circles tangent to a given circle

Pf

Consider the incidence correspondence

$$\begin{tikzcd} & \Phi := \{ (p, C) \in D \times \underset{\mathbb{P}^1 \times \mathbb{P}^3 \,\cong}{\mathcal{C}} \mid C \text{ tangent to } D \text{ at } p \} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathbb{P}^1 & & \mathbb{P}^3 \end{tikzcd}$$

in the case $p = o_\pm$, $m_p \geq 3$.

$$\pi_1^{-1}(p) = \{ C \mid \underbrace{m_p (C \cdot D) \geq 2}_{\text{2 linear conds}} \} \cong \mathbb{P}^1$$

$\rightsquigarrow$ Φ is a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$ $\Rightarrow$ irreducible.

Tonum & Sidhanth: Topologically this is $Bl_1 \mathbb{P}^1 \rightsquigarrow \mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$.

$\pi_2 : \Phi \dashrightarrow \mathbb{P}^3$ birational $\Rightarrow$ $Z_D = \pi_2(\Phi)$ is 2-dimensional

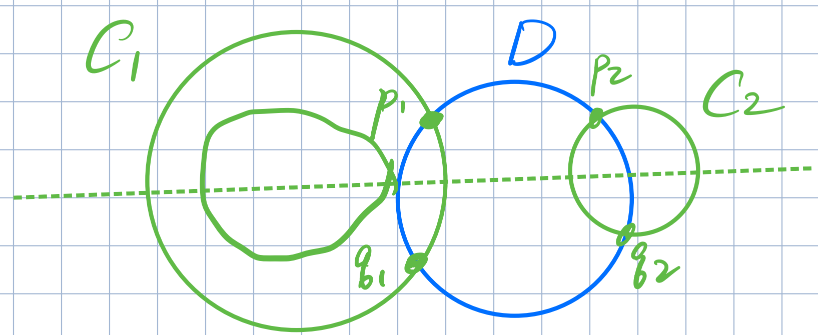
Circles tangent to a given circle

Pf Calculate $\deg Z_D$.

$\deg Z_D = \# Z_D \cap$ general line in $\mathcal{C} \cong \mathbb{P}^3$

$= \#$ circles in a general pencil tang to D.

Take general pencil $\overline{C_1 C_2}$ where $C_1 = Z[f]$, $C_2 = Z[g]$.



$\rightsquigarrow$ rational function $(f/g)|_D$ has 2 zeros $\{p_1, q_1\}$ and 2 poles $\{p_2, q_2\}$

$\Rightarrow$ Double cover $D \xrightarrow[\deg 2]{} \mathbb{P}^1$ branched along 2 points (R-H)

$= \deg Z_D$. $\square$

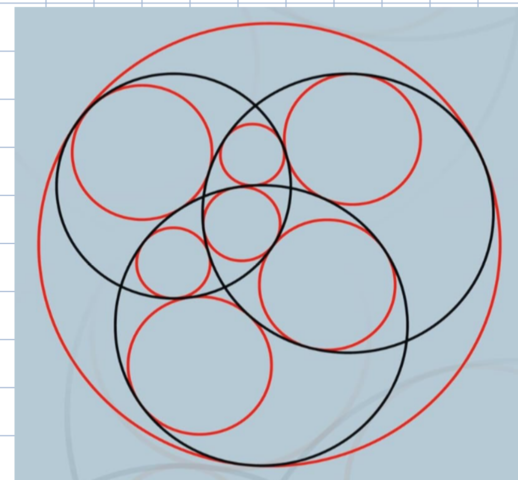
Circles of Apollonius

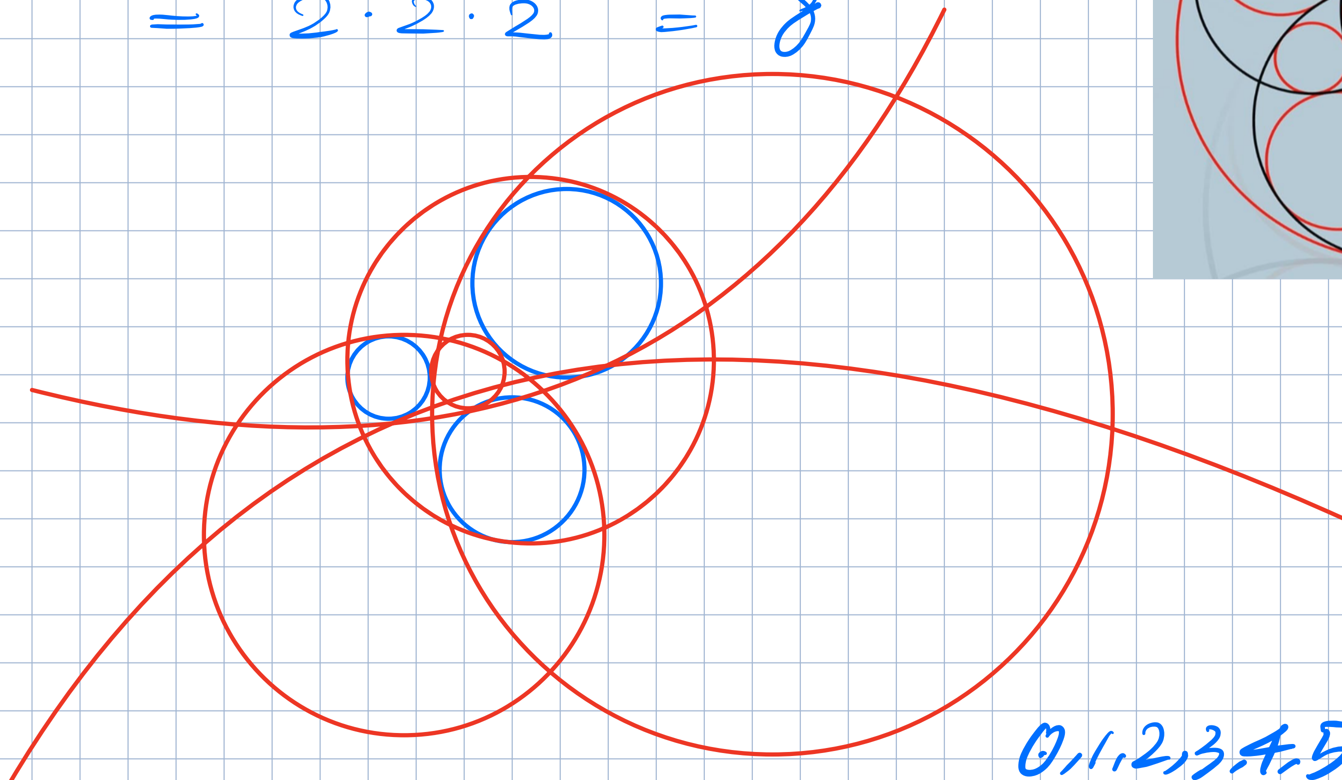
(By the argument from Steiner's Conics) $Z_D, Z_{D'}$ transverse for general $D, D' \in \ell$.

$\rightsquigarrow$ # circles tangent to 3 given ones

$= \# \; Z_{D_1} \cap Z_{D_2} \cap Z_{D_3}$

$= 2 \cdot 2 \cdot 2 = 8$





0, 1, 2, 3, 4, 5, 6, 8

Monodromy of Apollonian Circles

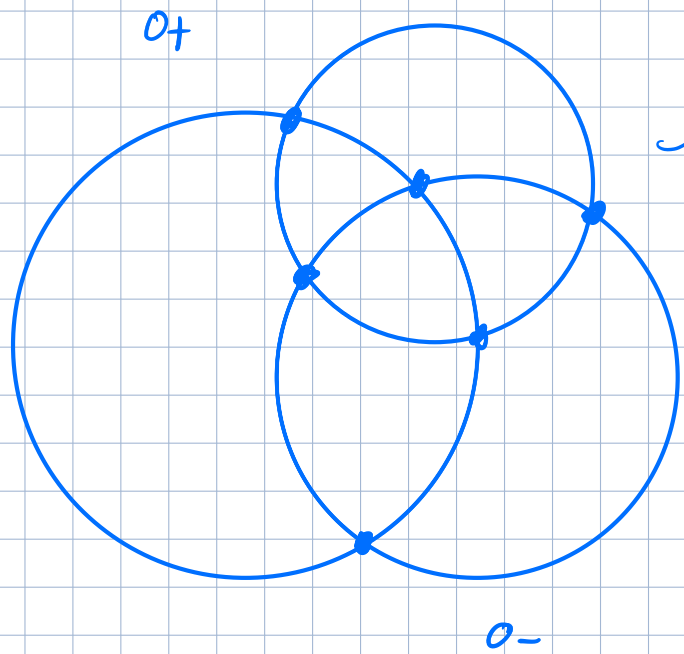
$$\{(\text{Triples of circles}, 8 \text{ tangent circles})\} \xrightarrow{8:1} \{\text{Triples of circles}\}$$

$\mathrm{Hilb}^r X$ hard when $r \geq 3$ $\dim X \geq 3$.

Problem ... This seems hard

Another approach to the circles of Apollonius, via the notion of *theta-characteristics*, is given in Harris [1982]. There is also an analogous notion of a *sphere* in $\mathbb{P}^3$; see for example Exercise 13.32.

A singular sextic



singular degree 6 plane curve

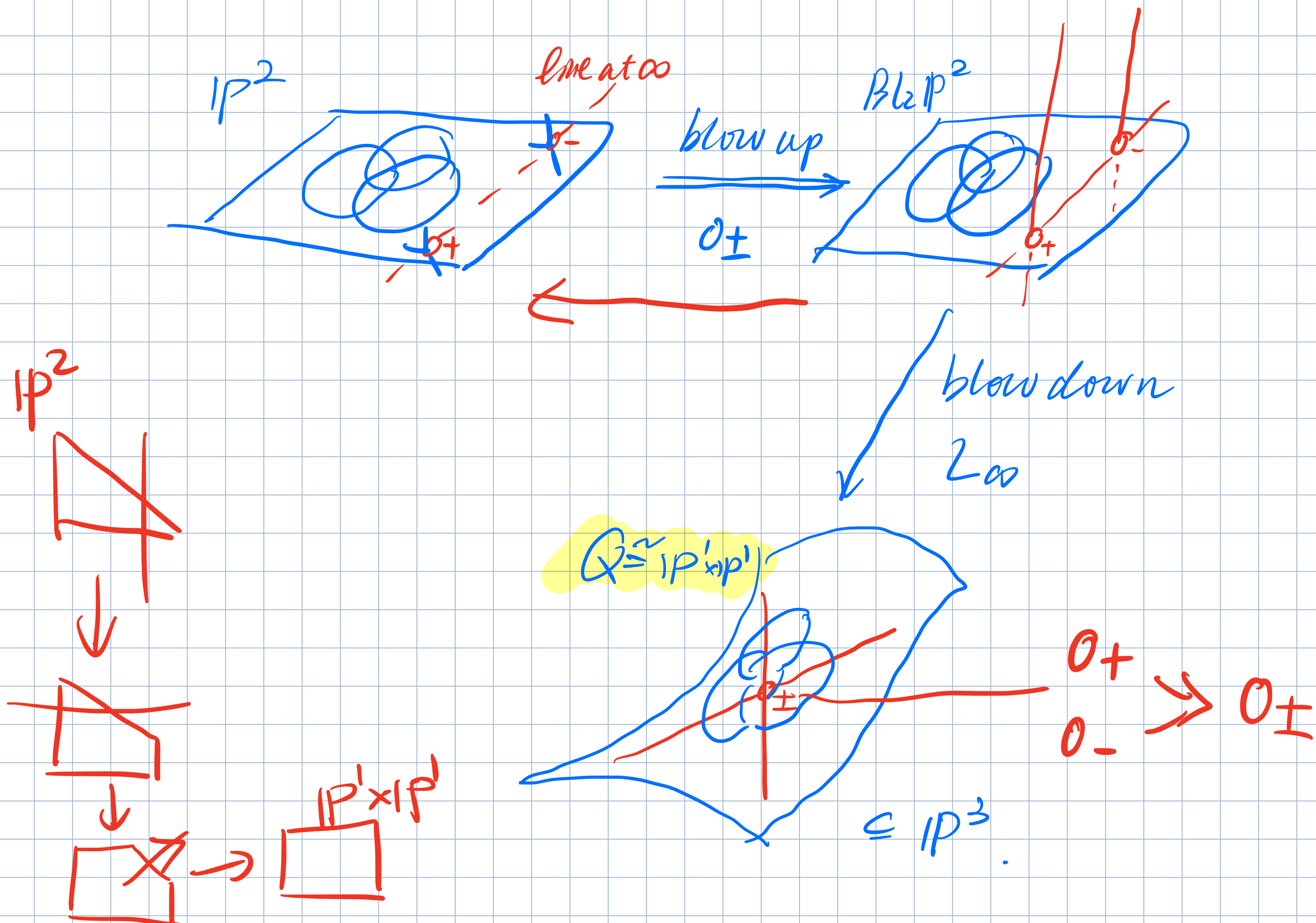
- triple nodes at $O_\pm$
- ordinary nodes at 6 points.

blow up at $O_\pm$

C: 6-nodal curve of genus $\frac{(6-1)(6-2)}{2} - 6 = 4$.

The canonical sextic in $\mathbb{P}^3$

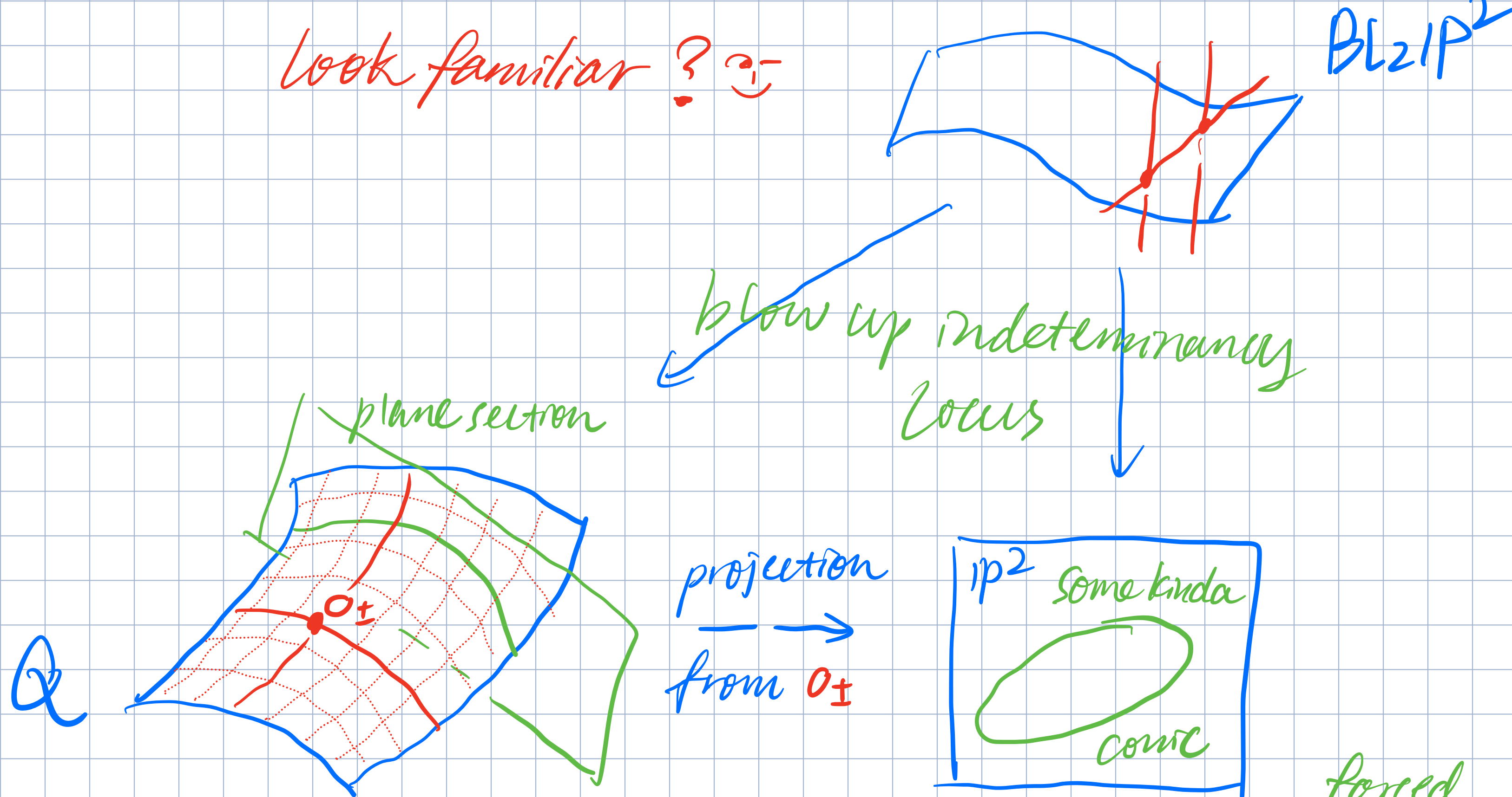
In fact: C is a canonical genus 4 curve in $\mathbb{P}^3$.

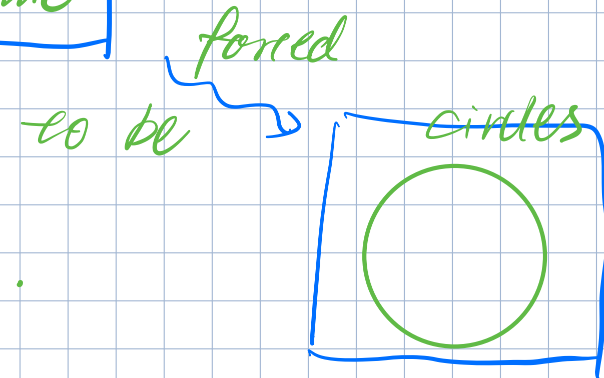


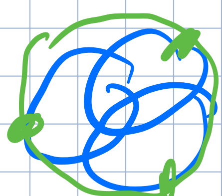
The canonical sextic in $\mathbb{P}^3$

Claim The hyperplane sections of Q are transforms of circles

Look familiar? :)

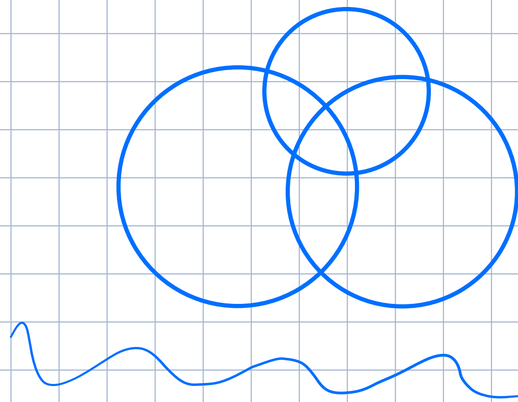




 $\longleftrightarrow$ odd thetas $\longleftrightarrow$ tritangents.

The canonical sextic in $\mathbb{P}^3$

Now, what's C?



$\sim 3H$

$\Rightarrow$ C is the complete intersection of $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$ with a completely reducible cubic

which is a ... canonical genus 4!

Canonical $\longrightarrow$ K_C is the hyperplane!

Circles of Apollonius = θ - Characteristics

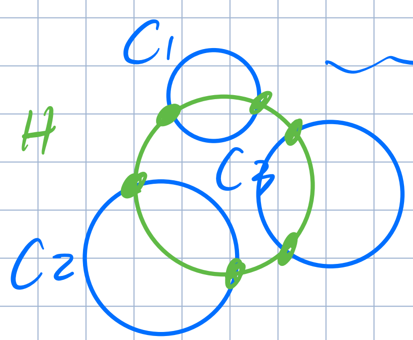
Recall: A theta-characteristic on C is a line bundle L s.t. $L^{\otimes 2} \cong \omega_C$

need to make sense of all of this!

Problem: C is singular!

A semicanonical divisor on C is D s.t. $2D \sim K_C$

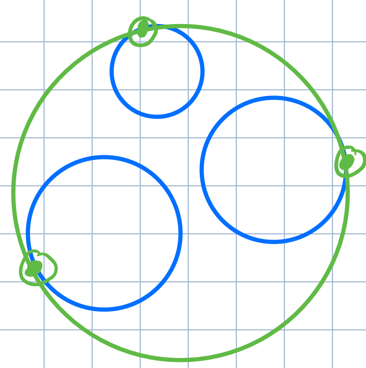
Consider: effective semicanonical divisors on C



$\rightsquigarrow H \cap C_i = 2\text{pts} \longrightarrow 2D \sim H \sim K_C$

$\Downarrow$

double points



A circle of Apollonius!

Circles of Apollonius = θ - Characteristics

Assuming for C :

Odd θ - characteristics $\xleftrightarrow{\sim}$ effective semicanonical divisors

Have :

① # circles of Apollonius = # odd θ-characteristics

$\overset{?}{=}$ 8 .

② Monodromy of Apollonian circles $\cong$ Symmetry group of S^-.

[Har '82]

Theta-Characteristics for Singular Curves!

Recall

C = smooth proper curve.

Theta-characteristic $\quad S(C) := \{L \mid L^{\otimes 2} \cong \omega_C\}$ $\left\{ \begin{array}{l} \text{odd } S^- \\ \text{even } S^+ \end{array} \right.$

2-torsion $\quad J_2(C) := \mathrm{Pic}^0(C)[2] \cong (\mathbb{Z}/2)^{2g}$.

Theta form $\quad \mathbb{F}_2$-quadratic form $q_L : J_2(C) \to \mathbb{F}_2$.

$$q_L(\omega) := h^0(L \otimes \omega) - h^0(L) \pmod 2$$

Weil pairing $\quad J_2(C) \cong H_1(C, \mathbb{Z}/2)/H_1(C, \mathbb{Z}) \cong H_1(C, \mathbb{Z}/2)$.

Intersection pairing mod 2 $\Rightarrow \lambda : J_2 \times J_2 \to \mathbb{F}_2$.

Riemann-Mumford relation

$$\lambda(w, \eta) = q_L(w+\eta) + q_L(w) + q_L(\eta)$$

Gorenstein Curves

Q: What's the minimal setting for theta-characteristics?

Want: locally free invertible sheaf $\mathcal{L}$

s.t. $\mathcal{L}^{\otimes 2} \simeq \omega_C$

what's this guy?

$\dim \mathrm{Hom}_R(k,R)=1$

$\mathrm{injdim}_R R < \infty$

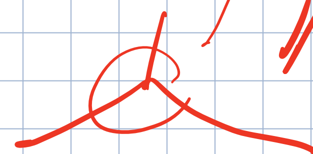
...

Def A scheme is Gorenstein if all local rings are Gorenstein rings.

Thm The dualizing sheaf ω_X is locally free if X is Gorenstein.

curves on smooth surface, etc.

Slogan: local complete intersections



$\mathbb{A}^3$

Ex/ Counterexample?

The Jacobian of a singular curve

Let C be reduced & Gorenstein.

Let $\tilde{C} \xrightarrow{\pi} C$ be the normalization.

$$0 \to \mathcal{O}_C^\times \to \pi_* \mathcal{O}_{\tilde{C}}^\times \to \mathcal{F} \to 0$$

$\oplus$ skyscraper sheaf at singular pts.

LES $\Longrightarrow$

$$0 \to \underbrace{H^0(\mathcal{O}_C^\times)}_{k^\times} \to \underbrace{H^0(\pi_* \mathcal{O}_{\tilde{C}}^\times)}_{k^\times} \to \underbrace{H^0(\mathcal{F})}_{\text{Something}}$$

$$\to \underbrace{H^1(\mathcal{O}_C^\times)}_{\mathrm{Pic}(C)} \to \underbrace{H^1(\pi_* \mathcal{O}_{\tilde{C}}^\times)}_{H^1(\tilde{C}, \mathcal{O}_{\tilde{C}}^\times) \cong \mathrm{Pic}(\tilde{C})} \to \underbrace{H^1(\mathcal{F})}_{0} \cdots$$

$\to 0$

The Jacobian of a singular curve

Let $\Gamma := H^0(\mathcal{F}) / \mathrm{Im}(H^0(\pi_* \mathcal{O}_{\tilde{C}}))$

Then from LES, ?

singular normalization θ ✓

$$0 \to \Gamma \to J(C) \xrightarrow{\pi^*} J(\tilde{C}) \to 0$$

so $\Gamma = \{ [L] \mid \pi^* L \text{ is trivial on } \tilde{C} \}$

2-torsion structure?? want θ-char ✓.

$$0 \to \Gamma_2 \to J_2(C) \xrightarrow{\pi^*} J_2(\tilde{C})$$

want know

The Jacobian of a singular curve

(2.2) Γ_2 is the nullspace of the bilinear form λ.

Pf It can be shown from the definition that

$$\lambda(\omega, \eta) = \lambda(\pi^*\omega, \pi^*\eta)$$

from which we see that $\pi^*\omega = \pi^*\eta = 0$. $\square$

(2.3) The restriction $q_L|_{\Gamma_2}$ is a linear functional

(2.4) ... is independent of choice of L.

Def The Linear part of the theta form is $\ell := q_L|_{\Gamma_2} : \Gamma_2 \to \mathbb{F}_2$.

Counting θ-characteristics on singular curves

$\tilde{g} :=$ sum of genera of components of $\tilde{C}$.

$k :=$ $\operatorname{rank}_{\mathbb{F}_2} \Gamma_2$

(2.7) If the linear part $\ell \neq 0$, then

$$\# S^+ = \# S^- = \frac{1}{2} \# S = 2^{2\tilde{g}+k-1}$$

(2.8) If $\ell = 0$, then

either $\quad \# S^+ = 2^{\tilde{g}+k-1}(2^{\tilde{g}}+1) \qquad \# S^- = 2^{\tilde{g}+k-1}(2^{\tilde{g}}-1)$

or $\qquad$ the other way around.

Counting θ-characteristics on singular curves

Proof of 2.7

Key: By definition of the linear part,

for any θ-characteristic L and $\omega \in \Gamma_2$, we have

$$h^0(L \otimes \omega) \equiv l(\omega) + h^0(L) \pmod 2 .$$

Consider fibers of $J_2(C) \xrightarrow{\pi^*} J_2(\tilde{C})$:

$$(\pi^*)^{-1}(L) = \{ L' \in J_2(C) \mid \pi^* L' = L \}$$

$$= \text{coset } L_0 + \Gamma_2$$

$\Rightarrow$ half even, half odd.

$$\Rightarrow \# S^+ = \# S^- = \frac{1}{2} \# S = 2^{2\tilde{g}+k-1}$$

$\square$

The structure of the group Γ_2

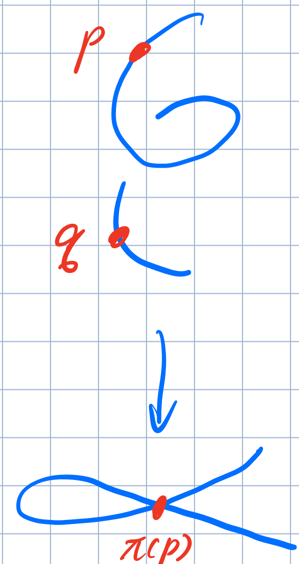
Q: What is k??

$\Gamma_2 = \{$ 2-torsion line bdls on C that become trivial on $\tilde{C} \}$.

What's such a line bundle?

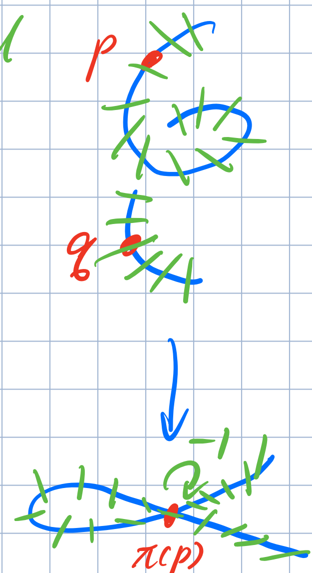
Eg 1 Twisted cubic $\tilde{C}$

nodal cubic C



Trivial line bdl $\mathcal{O}_{\tilde{C}}$

glue via (-1).



The structure of Γ_2

Claim All elements of Γ_2 arise this way!

Def For the normalization $\tilde{C} \to C$, let $p_1, \dots, p_n$ be the preimages of a node p.

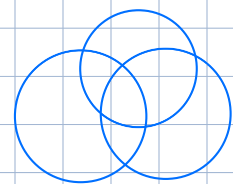
Define L_{p_i} to be the line bundle obtained from gluing $\mathcal{O}_{\tilde{C}}$ with a (-1)-twist between p_i and other p_j.

Then:

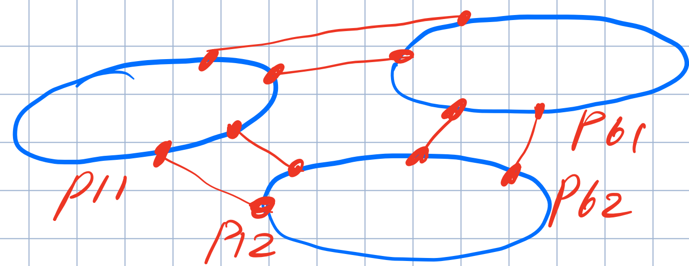
Thm Γ_2 has generators $e_{p_i} := [L_{p_i}]$ for all singular points p

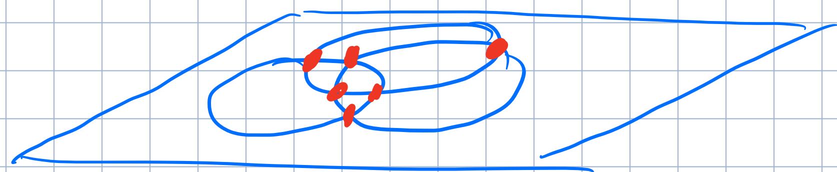
and relations
- at p: $\sum e_{p_i} = 0$
- at irred. component $\tilde{C}_j$: $\sum_{q \in \tilde{C}_j} e_q = 0$

Circles of Apollonius = θ-Characteristics

Recall that the sextic C =  has 6 nodes,

and $\tilde{C}$ = disjoint union of 3 rational curves





$\rightsquigarrow$ 12 generators $\rightsquigarrow$ For all i, $p_{i1}+p_{i2}=0$

$\longrightarrow$ The 4 pts on each branch sum up to 0 4 independent generators

Circles of Apollonius = θ-Characteristics

So: $\#$ Apollonian circles

$= \#$ odd θ-characteristics

$= 2^{4+0-1}$

$= 8.$

Analyzing the action of Γ on S^-

GAP 32,49

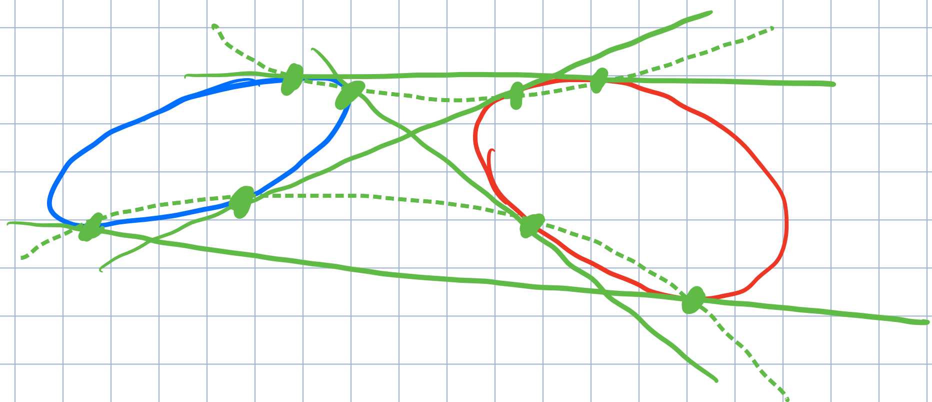
$\rightsquigarrow$ Monodromy $\cong$ inner holomorph of D_8

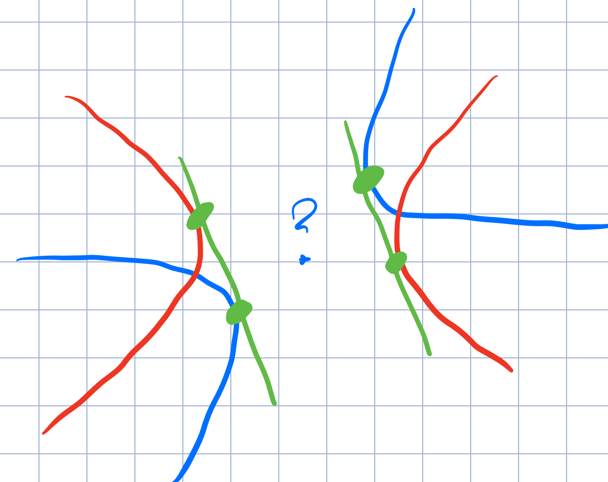
$\cong D_8 \rtimes \mathrm{Inn}(D_8)$

$$\begin{pmatrix} 1 & * & * & * \\ & 1 & & * \\ & & 1 & * \\ & & & 1 \end{pmatrix}$$

Other amusing applications

Theorem (3.3). *Let $C_1, C_2 \subset \mathbf{P}^2$ be plane conics meeting transversely, $L_1, \ldots, L_4$ their four common tangent lines. Then the eight points $\{p_j^i = L_i \cap C_j\}$ of tangency lie on a conic.*



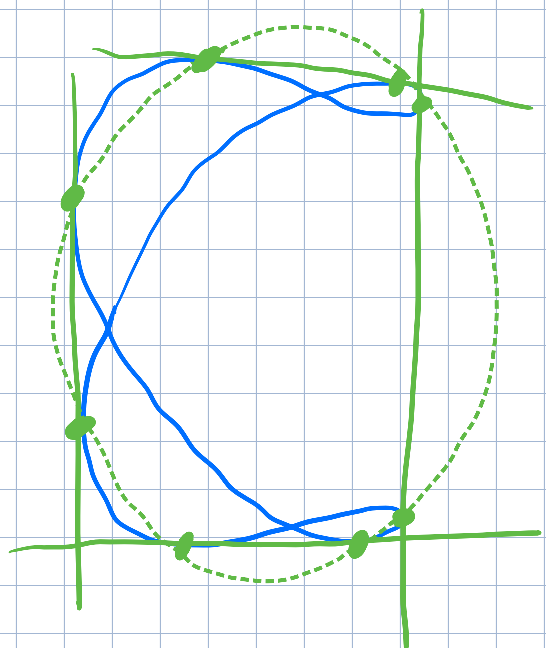


pf Syzygetic quadruples of odd theta characteristics on a (2,2)-type singular plane quartic.

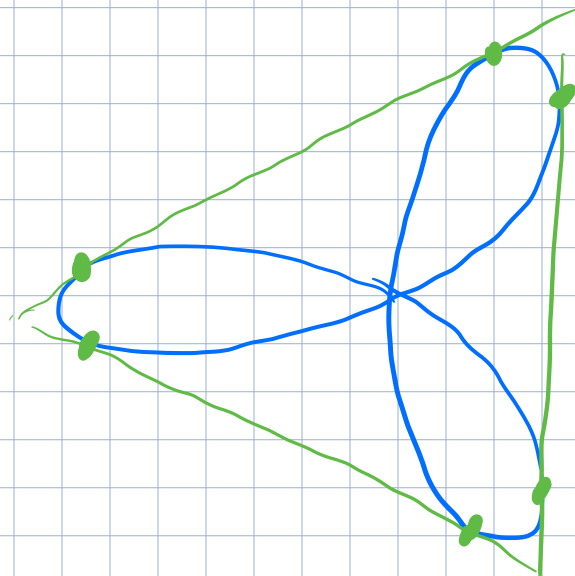
Other amusing applications

Theorem (3.4) (cf. [**10**]). *If $C \subset \mathbf{P}^2$ is an irreducible quartic with 3 nodes, then the eight points of contact of C with its four bitangent lines all lie on a conic.*

Ampersand curve



Trifolium curve (3-leafed clover)



1 imaginary bitangent ?

Theorem (3.5) (cf. [**10**]). *An irreducible quartic with a triple point has four bitangent lines, whose eight points of contact lie on a conic.*

Other amusing applications

THEOREM (3.8). *If C is a plane quartic with an ordinary node and a cusp, C has exactly 4 bitangents; and their 8 points of contact do not lie on a conic.*

singularities	$g = g(C)$	$k = \mathrm{rk}\,\Gamma_2$	l	$q(C)$	$\#S$	$\#S^-$
smooth	3	0	0	0	64	28
one node	2	1	$\neq 0$	–	32	16
one cusp	2	0	0	1	16	10
two nodes	1	2	$\neq 0$	–	16	8
tacnode	1	1	0	1	8	6
one node; one cusp	1	1	$\neq 0$	–	8	4
two cusps	1	0	0	0	4	1
ramphoid cusp	1	0	0	1	4	3
three nodes	0	3	$\neq 0$	–	8	4
one node; one tacnode	0	2	$\neq 0$	–	4	2
one oscnode	0	1	$\neq 0$	–	2	1
two nodes; one cusp	0	2	$\neq 0$	–	4	2
one tacnode; one cusp	0	1	0	0	2	0
one node; two cusps	0	1	$\neq 0$	–	2	1
one node, one ramphoid cusp	0	1	$\neq 0$	–	2	1
three cusps	0	0	0	1	1	1
one cusp, one ramphoid cusp	0	0	0	0	1	0
one hyper-ramphoid cusp	0	0	0	0	1	0
one ordinary triple point	0	2	0	1	4	4
one triple point, two branches	0	1	0	1	2	2
one unibranch triple point	0	0	0	1	1	1

Moduli-theoretic meaning?

Theta-Characteristics and Their Moduli [Far '12]

Def A reduced, connected, nodal curve C is quasistable if for any rational component E,

(i) $k_E = \# E \cap \overline{C-E} \geq 2$.

(ii) For all E, E' s.t. $k_E, k_{E'} = 2$, $E \cap E' = \emptyset$.

exceptional component

Def (theorem-defn) A stable spin curve of genus g is (C, θ, β)

- C: quasistable of arithmetic genus g
- $\theta \in \mathrm{Pic}^{g-1}(C)$, $\theta|_E \cong \mathcal{O}_E(1)$ for all exceptional comp E.
- $\beta: \theta^{\otimes 2} \longrightarrow \omega_C$ sheaf hom s.t. $\beta|_E \neq 0$ for all exp. E.

The moduli stack of spin curves $\overline{S}_g$ parametrizes ↺

$$\overline{S}_g = \overline{S}_g^+ \sqcup \overline{S}_g^-$$

$$\begin{tikzcd} S_g \arrow[r] \arrow[d, "2^{2g}:1"'] \arrow[dr, phantom, "\square"] & \overline{S}_g \arrow[d, "2^{2g}:1"] \\ M_g \arrow[r] & \overline{M}_g \end{tikzcd}$$

Theta-Characteristics and Their Moduli [Far '12]

Theorem 0.1. *The birational type of the moduli spaces $\overline{\mathcal{S}}_g^+$ and $\overline{\mathcal{S}}_g^-$ of even and odd spin curves of genus g can be summarized as follows:*

$$\overline{\mathcal{S}}_g^+: \begin{array}{c|c} g > 8 & \text{general type} \\ \hline g = 8 & \text{Calabi-Yau} \\ \hline g \leq 7 & \text{unirational} \end{array} \qquad \overline{\mathcal{S}}_g^-: \begin{array}{c|c} g \geq 12 & \text{general type} \\ \hline g \leq 8 & \text{unirational} \\ \hline 9 \leq g \leq 11 & \text{uniruled} \end{array}$$

Bibliography

Eisenbud & Harris 2016. 3264 and All That

Harris 1982. Theta-Characteristics on Algebraic Curves

Farkas 2012. Theta-characteristics and their Moduli