

28 Presentation 5 (December 6, 2024): Tits construction, Freudenthal Magic Square, and Exceptional Lie Algebras

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28.1 Composition Algebras

Recall that a *quadratic form* on a k -vector space V is a map $n : V \rightarrow k$ such that $n(av) = a^2n(v)$ for $a \in k$ and $v \in V$, and such that

$$n(x, y) = n(x + y) - n(x) - n(y)$$

is bilinear. Such a form is called *nondegenerate* if its associated bilinear form is.

28.1.1 Definition. A *composition algebra* is a not necessarily associative algebra C with a non-degenerate quadratic form n , called its *norm*, which admits composition. That is, such that

$$n(xy) = n(x)n(y)$$

for all $x, y \in C$.

It is easy to check that $n(1) = 1$, and so we can define for any composition algebra C its associated *trace*

$$t_C(x) = n(x, 1) = n(x + 1) - n(x) - 1.$$

Given such a trace, there is a natural involution

$$x \mapsto \bar{x} = n(x, 1) - x$$

such that $x + \bar{x} = t_C(x)$ and $x\bar{x} = n(x)$ (see Schafer for the details).

Over $\mathbb{C}$ there is a particularly simple classification of such algebras:

28.1.2 Theorem (Hurwitz). *Up to isomorphism there are only four composition algebras over $\mathbb{C}$, of dimensions 1, 2, 4, 8 respectively. They are*

1. $\mathbb{C}$ with norm $n(x) = x^2$
2. $\mathbb{C} \times \mathbb{C}$ with norm $n((a, b)) = ab$
3. $M_2(\mathbb{C})$ with norm given by the determinant
4. The split Cayley algebra $C_s(\mathbb{C})$.

In dimensions 1, 4, 8 these are simple, but the dimension 2 algebra is only semi-simple. The split Cayley algebra is perhaps best thought of as the algebra $M_2(\mathbb{C}) \times M_2(\mathbb{C})$ with product given by

$$(A, B)(C, D) = (AC + \overline{DB}, B\overline{C} + DA).$$

where the involution is defined by

$$\overline{\begin{bmatrix} a & b \\ c & d \end{bmatrix}} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

28.1.3 Remark. The construction of the split Cayley algebra in this way is via the Cayley-Dickson construction. It turns out that all composition algebras can be created through this process.

For future reference we will work out the trace zero elements of each composition algebra. In dimension 1 this is the trivial subspace. In dimension 2, the trace is given by

$$t_C(a, b) = a + b$$

so the trace zero elements are $\mathbb{C}(1, -1)$.

In dimension 4, the trace is given by

$$t_C \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = (a + d)I,$$

so the trace zero elements are just the trace zero matrices. This space has dimension 3.

In dimension 8, the trace is given by

$$t_C(A, B) = (A, B) + \overline{(A, B)} = (A, B) + (\bar{A}, -B) = (A + \bar{A}, 0)$$

so the trace zero elements are those pairs with a trace zero matrix as the first component. This space has dimension 7.

The last thing we will need are the derivations of these composition algebras. In particular, we are concerned with the *inner derivations*, which are the derivations D which lie in the Lie algebra generated by the right and left multiplication operators under the commutator bracket:

$$L_x(y) = xy, \quad R_x(y) = yx.$$

It turns out that composition algebras are a part of a larger collection of algebras called *alternative algebras*, which are algebras which satisfy the identity

$$x(xy) = (xx)y.$$

A rather tedious calculation shows that the inner derivations for any such algebra are of the form

$$L_x - R_x + D_{y,z}$$

where

$$D_{y,z} = L_{[y,z]} - R_{[y,z]} + 3[L_y, R_z].$$

To make this more comprehensible, observe that

$$D_{y,z}(a) = [[y, z], a] + 3(y, a, z)$$

where $(y, a, z) = y(az) - (ya)z$ is the associator.

In dimension 1, 2 are composition algebras are commutative, so $L_x - R_y$ will act as zero. The dimension 4 algebra is associative, so the associative $[L_y, R_z]$ will always vanish in $D_{y,z}$. Since the dimension 4 algebra is also simple, it follows that every

element can be expressed as a commutator, so $L_x - R_x = D_{y,z}$ for some y, z . We surmise then that all inner derivations for composition algebras of dimension 1, 2, 4 over $\mathbb{C}$ are of the form $D_{y,z}$. It turns out this result extends to dimension 8 as well, but the proof is not so simple.

28.1.4 Remark. It turns out that over $\mathbb{C}$, all derivations of composition algebras are inner derivations, so this restriction turns out to be no restriction at all.

28.2 Jordan Algebras

28.2.1 Definition. A *Jordan algebra* J is a nonassociative algebra such that for all $x, y \in J$ we have

- $xy = yx$ (i.e. the algebra is commutative)
- $(xy)(xx) = x(y(xx))$ (the Jordan identity)

28.2.2 Remark. These axioms imply that x^n is well-defined: no matter how you parenthesize the product of n x s, you get the same product. They also imply that $x^m(x^n y) = x^n(x^m y)$. These properties were the original motivation for defining Jordan algebras.

28.3 Example. Let A be any associative algebra. Define the *Jordan product*

$$x \circ y = \frac{xy + yx}{2}.$$

Then A with the Jordan product is no longer associative, but it is a Jordan algebra.

28.4 Example. Let $H_n(A)$ be the set of $n \times n$ Hermitian matrices over A . This set is not closed under matrix multiplication, but we have (for $x, y \in H_n(A)$)

$$(x \circ y)^* = \frac{1}{2}((xy)^* + (yx)^*) = \frac{1}{2}(y^* x^* + x^* y^*) = x \circ y$$

so $H_n(A)$ is closed under the Jordan product. Thus $H_n(A)$ is a Jordan Algebra: one that we'll need soon for the Tits construction!

We shall restrict our attention to unital Jordan algebras J , with a “normalized trace” $t_J : J \rightarrow \mathbb{C}$. This is a $\mathbb{C}$ -linear map such that $t_J(1) = 1$ and $t_J((xy)z) = t_J(x(yz))$ for all $x, y, z \in J$.

28.5 Example. In the case of $J = H_n(A)$, we have a normalized trace $t_J = \frac{1}{n} \text{Tr}$.

We shall define J^0 to be the set $\{x \in J : t_J(x) = 0\}$. Note that for any $x \in J$, we have $x - t_J(x) \in J^0$. Thus we have a decomposition $J = \mathbb{C}1 \oplus J^0$.

Let $x \star y = xy - t_J(xy)1$, the part of xy in J^0 . This defines a commutative multiplication on J^0 , which shall be used in the Tits construction.

Finally, let $d_{x,y} : J \rightarrow J$ be the function $d_{x,y}(z) = x(yz) - y(xz)$. This is the inner derivation of J determined by x, y . Our sources claim that the Jordan algebra axioms implies that this is a derivation (satisfying the Leibniz rule), but we could not reproduce that calculation.

We have $d_{1,x}(z) = x(z) - (xz) = 0$, so $d_{1,x} = 0$ for all $x \in J$. Thus we have the space of inner derivations given by

$$\text{in}\text{der} J = \{d_{x,y} : x, y \in J^0\}.$$

Thus we have all the pieces of the Tits construction.

28.3 Tits Construction and the Freudenthal Magic Square

Throughout, let $\mathbb{F}$ be an algebraically closed field of characteristic not 3. It is a fact that there are exactly 4 non-isomorphic finite-dimensional normed algebras over $\mathbb{F}$, one each in dimensions 1, 2, 4, and 8. As a vector space, each of these algebras is a direct sum of two copies of the previous. The reason this “doubling” process stops at dimension 8 is because the construction requires the previous algebra to be associative. However, the 8 dimensional normed algebra always ceases to be associative, instead being alternative, a weaker notion.

The discovery of the Tits construction and the Tits-Freudenthal magic square thereupon corroborates the folklore assertion that *all* exceptional Lie algebras owe their existence to the Cayley octonions (over $\mathbb{F}$, the unique normed algebra of dimension 8).

Cartan (Élie, the father) first discovered that G_2 arises from the automorphisms of $\mathbb{O}$.

28.3.1 Theorem. [Cartan 1914]

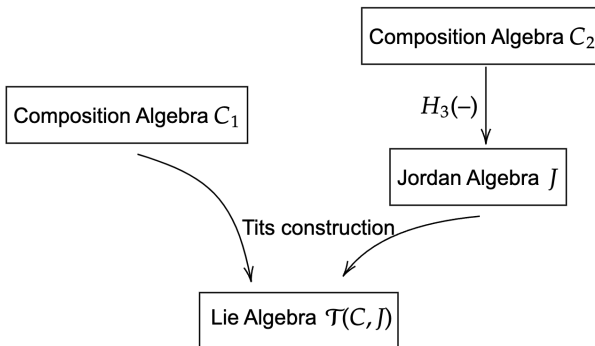
$$\text{Aut } \mathbb{O} \cong G_2, \quad \text{der } \mathbb{O} \cong \mathfrak{g}_2.$$

Next, recall that $n \times n$ Hermitian matrices over a composition algebra has the structure of a Jordan algebra. In particular, take $n = 3$. The Jordan algebra $H_3(C)$ is sometimes referred to as the *Freudenthal algebra*, the automorphisms of which gives rise to F_4 .

28.3.2 Theorem. [Chevalley-Schafer 1950]

$$\text{Aut}(H_3(\mathbb{O})) \cong F_4, \quad \text{der}(H_3(\mathbb{O})) \cong \mathfrak{f}_4.$$

Tits in 1962 discovered a construction which takes in a pair of composition algebras, or more generally, a composition algebra and a Jordan algebra, and results in a Lie algebra:



Let C be a composition algebra with t_C its trace, D its associated derivation ($D_{a,b}(c) = [[a, b], c] + 3(a, c, b)$). Let J be a Jordan algebra with t_J its normalized trace, $*$ the product on J^0 , and d its associated derivation ($d_{x,y}(z) = x(yz) - y(xz)$). The Tits construction on C and J has the underlying vector space

$$\mathcal{T}(C, J) := \text{in}\mathcal{D} C \oplus (C^0 \otimes J^0) \oplus \text{in}\mathcal{D} J.$$

It admits the structure of a Lie algebra in the following way:

- $\text{in}\mathcal{D} C$ and $\text{in}\mathcal{D} J$ are Lie subalgebras.
- $[\text{in}\mathcal{D} C, \text{in}\mathcal{D} J] = 0$.

Let a, b denote elements of C^0 , and x, y elements of J^0 . Let D and d denote elements of $\text{in}\mathcal{D} C$ and $\text{in}\mathcal{D} J$, *resp.* Then,

- $[D, a \otimes x] = D(a) \otimes x$. $[d, a \otimes x] = a \otimes d(x)$.
- $[a \otimes x, b \otimes y] = t_J(xy)d_{a,b} + [a, b] \otimes (x * y) + 2t_C(ab)d_{x,y}$.

One should check that this endows $\mathcal{T}(C, J)$ with the structure of a Lie algebra.

Take C_1 to be the composition algebra of dimensions 1, 2, 4, or 8 over $\mathbb{F}$. Take J to be H_3 over the four composition algebras. The Tits construction outputs the **Freudenthal Magic Square** of Lie algebras:

	1	2	4	8
1	A_1	A_2	C_3	F_4
2	A_2	$A_2 \times A_2$	A_5	E_6
4	C_3	A_5	D_6	E_7
8	F_4	E_6	E_7	E_8

Table 1: The Freudenthal Magic Square

One immediately notices the symmetry of this table across the main diagonal. This fact in addition to the appearance of many of the exceptional algebras is what earned this table the word “magic” in its name.

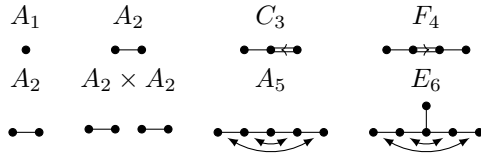
One can further extend this table by letting J be one of the smaller dimensional Jordan algebras over $\mathbb{F}$:

	$\mathbb{F}$	$\mathbb{F} \times \mathbb{F} \times \mathbb{F}$	1	2	4	8
1	0	0	A_1	A_2	C_3	F_4
2	0	$\mathfrak{U}$	A_2	$A_2 \times A_2$	A_5	E_6
4	A_1	$A_1 \times A_1 \times A_1$	C_3	A_5	D_6	E_7
8	G_2	D_4	F_4	E_6	E_7	E_8

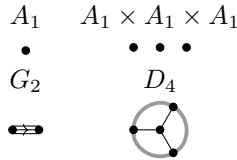
Table 2: The extended Freudenthal Magic Square

All exceptional types, including G_2 , appear in the extended table. However, the table fails to be symmetric. In fact, one may further extend the table to a *magic triangle*, so that it is again symmetric — cf. [Deligne & Gross 2002] and [Cvitanović 2007]. The appearance of D_4 in the extended table, in addition to its 3-fold symmetry, is the reason why some consider it as one of the exceptionals.

More symmetries appear in Freudenthal's magic square. The first row can be obtained from the second row from a $\mathbb{Z}/2$ -diagram folding:



The first column in the extended table can be similarly obtained from the second column by a $\mathbb{Z}/3$ -folding.



Vinberg in 1966 discovered a different construction using triality algebras, for which the diagonal symmetry of the 4×4 magic square is obvious. For a normed algebra A , the *triality group* of A is

$$\mathrm{Tri}(A) : \{(U_1, U_2, U_3) \in SO_3(A) \mid U_1(xy) = U_2(x)U_3(y)\}.$$

$\mathrm{Tri}(A)$ is an algebraic group, and we may take its associated Lie algebra $\mathfrak{tri}(A)$. For a pair of composition algebras A and B , let A_i and B_i denote different but isomorphic copies of A , *resp.* B . Vinberg's construction goes as follows:

$$\mathfrak{g}(A, B) := \mathfrak{tri} A \otimes \mathfrak{tri} B \oplus A_1 \otimes B_1 \oplus A_2 \otimes B_2 \oplus A_3 \otimes B_3.$$

The Lie algebra structure is given as follows:

- $\mathfrak{tri} A \times \mathfrak{tri} B$ is a Lie subalgebra.
- Let $a_i \in A_i$ and $b_i \in B_i$. Set $[a_1 \otimes b_1, a_2 \otimes b_2] = a_1 a_2 \otimes b_1 b_2$, viewed as an element of $A_3 \otimes B_3$.

Vinberg's construction is obviously symmetric in A and B .

What are these triality algebras? We have $0, \mathbb{F} \times \mathbb{F}, \mathfrak{sl}_2 \times \mathfrak{sl}_2 \times \mathfrak{sl}_2, \mathfrak{so}_8$ corresponding to the normed algebras of dimensions 1, 2, 4, and 8. The name triality comes from the 3-fold symmetry of the D_4 diagram.

Some amazing projective geometry arises from the Freudenthal magic square, cf. [Landsberg & Manivel 2001].

28.4 Construction of Exceptional Algebras

A few final remarks on the exceptional Lie algebras over $\mathbb{C}$ that arise in this way from the Tits construction. Throughout, let $J = H_3(C_s(\mathbb{C}))$.

Looking in the first row of the magic square, we first see that in the case where $C = \mathbb{C}$, we see that $C^0 = 0$ and $\text{in}\mathbf{der}(\mathbb{C}) = 0$. Then, $\mathfrak{f}_4 = \text{in}\mathbf{der}(J)$. A theorem of Chevalley and Schafer ([Theorem 28.3.2](#)) shows that $\mathfrak{f}_4$ is simple and that the complex dimension of $\mathfrak{f}_4$, $\dim_{\mathbb{C}} \mathfrak{f}_4 = 52$. An account of this argument can be found in [\[Schafer 1966\]](#) as Theorem 4.9. Roughly, Schafer argues for a certain decomposition of $\mathfrak{f}_4$ as a 28-dimensional part with three 8-dimensional parts, for a total dimension of 52, and shows that this algebra is simple. In this way one can conclude that the inner derivations of $H_3(C_s(\mathbb{C}))$ in fact span all derivations of $H_3(C_s(\mathbb{C}))$, since this is a subalgebra of $\mathfrak{f}_4$ and $\mathfrak{f}_4$ is simple.

Next, in the case where $C = \mathbb{C} \times \mathbb{C}$, $C^0 = \mathbb{C}(1, -1) \cong \mathbb{C}$ and $\text{in}\mathbf{der}(C) = 0$ since C is still commutative and associative. Then, we see that $\mathfrak{e}_6 = \mathbb{C} \otimes_{\mathbb{C}} J^0 \oplus \text{in}\mathbf{der}(J) = J^0 \oplus \text{in}\mathbf{der}(J)$. Note that $\dim_{\mathbb{C}} J = 27$ since we can have any element of $C_s(\mathbb{C})$ in the upper-triangular off-diagonal entries, but we must have an element of $\mathbb{C}$ in the three diagonal entries, for a total complex dimension of $3 \cdot 8 + 3 \cdot 1 = 27$. Then, $\dim_{\mathbb{C}} J^0 = 26$, so $\dim_{\mathbb{C}} \mathfrak{e}_6 = 26 + 52 = 78$. An argument for showing that $\mathfrak{e}_6$ constructed in this way is simple is given in [\[Schafer 1966\]](#) as Theorem 4.12. Note that $\mathfrak{e}_6$ is not the only 78-dimensional simple complex Lie algebra, as Lie algebras of type B_6 and C_6 are also 78-dimensional. To show that this Lie algebra is not isomorphic to either of these two, note that $\mathfrak{e}_6$ constructed in this way acts on J irreducibly, so $\mathfrak{e}_6$ has an irreducible 27-dimensional complex representation. One can verify (with highest weight theory) that neither B_6 nor C_6 admit an irreducible representation of this dimension, so $\mathfrak{e}_6$ is not isomorphic to either of these other Lie algebras.

In the third row of our square where $C = M_2(\mathbb{C})$, $C^0 = \mathfrak{sl}_2(\mathbb{C})$, and $\text{in}\mathbf{der}(C) = \{\text{ad}_{[a,b]} : a, b \in M_2(\mathbb{C})\}$. One can check that $\text{in}\mathbf{der}(C) \cong \mathfrak{sl}_2(\mathbb{C})$ – as a good heuristic, note that for any two matrices $a, b \in M_2(\mathbb{C})$, $[a, b] \in \mathfrak{sl}_2(\mathbb{C})$. Then, one can note that our $\mathfrak{e}_7 \cong \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C}) \otimes J^0 \oplus \text{in}\mathbf{der}(J)$, which can be rewritten at $\mathfrak{sl}_2(\mathbb{C}) \otimes (\mathbb{C} \oplus J^0) \oplus \text{in}\mathbf{der}(J) \cong \mathfrak{sl}_2(\mathbb{C}) \otimes J \oplus \text{in}\mathbf{der}(J)$ (as vector spaces – the bracket structure from the Tits construction may not be preserved by this rearrangement). Here, $\dim_{\mathbb{C}} \mathfrak{e}_7 = 3 \cdot 27 + 52 = 81 + 52 = 133$. As a remark, the third row of the magic square is related to the Kantor-Koecher-Tits construction of Lie algebras from a Jordan algebra, from which $\mathfrak{e}_7$ can also be obtained from the Jordan algebra $H_3(C_s(\mathbb{C}))$.

Finally, in the bottom-right corner, we can analyze the largest case of our construction when $C = C_s(\mathbb{C})$. Here, we realize $\mathfrak{e}_8 = \text{in}\mathbf{der}(C_s(\mathbb{C})) \oplus C_s(\mathbb{C})^0 \otimes J^0 \oplus \text{in}\mathbf{der}(J)$. Recall that by [Theorem 28.3.1](#), $\text{in}\mathbf{der}(C) = \mathfrak{g}_2$, and from the construction of $\mathfrak{g}_2$ as presented in Section 27, we know $\dim_{\mathbb{C}} \mathfrak{g}_2 = 14$. Recall that the trace zero elements of $C_s(\mathbb{C})$ have dimension 7, so we can compute $\dim_{\mathbb{C}} \mathfrak{e}_8 = 14 + 7 \cdot 26 + 52 = 248$.

This analysis gives us the dimensions of the five complex exceptional Lie algebras as 14, 52, 78, 133, and 248, corresponding to $\mathfrak{g}_2, \mathfrak{f}_4, \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8$, respectively. Their Weyl groups are also correspondingly large ($\mathfrak{f}_4$'s is the smallest with 1152 elements) and relate to other geometries from which these Lie algebras arise.