Rational Singularities of Complex Surfaces

a word-by-word translation

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Introduction

The main result of this work is the following theorem: There is only one non-regular two-dimensional analytic local ring which is factorial. This factorial local ring is the local ring of the singular point in the quotient space formed by the action of the binary icosahedral group on the komplex plane. Following KLEIN, this local ring is $\mathbb{C}\{x,y,z\}/\left(x^2+y^3+z^5\right)$. That this local ring is factorial was shown by MUMFORD in his work about the topology of normal singularities of algebraic surfaces. I will also use this result from the work of MUMFORD, in order to show the uniqueness of this factorial ring in §3. Moreover for that, I will need the results of ARTIN about rational singularities, which will be referred to in §1.

The importance of rational singularities in this context arises from the fact that they are precisely the two-dimensional normal singularities with a near-factorial local ring. There are a number of recent investigations about near-factorial and factorial local rings – for near-factorial local rings, e.g. the works of KOESTNER and STORCH [20, 29], for factorial local rings a few works of SAMUEL, e.g. [24], and SCHEJA [25,26]. There already, one also finds special cases of the theorem about the uniqueness of the icosahedral singularity.

The icosahedral singularity belongs to a special class of singularities, namely the singularities of quotient spaces which arise from the action of a proper discontinuous group on a two-dimensional complex manifold. In §2, I have completely classified these singularities under application of the results of HIRZEBRUCH, MUMFORD, and PRILL. The classification is only an étude, according to the general results of PRILL about quotients of complex spaces [22]. Nevertheless is it, apart from its importance for the proof of the main theorem, also interesting for a few reasons:

- Firstly, it represents a systematic summary of the results of numerous investigations by GODEAUX on quotient singularities.
- Secondly, the connection between the analytic structure of these singularities and their many-times investigated topological structure is clarified.
- Thirdly, it results immediately from the classification that the singularities of quotient spaces from 2-dimensonal complex manifolds are rigid.

I have already derived one special case of this result, the rigidity of rational double points, from a theorem of KIRBY in [4], and TJURINA has recently proven the rigidity of certain other rational singularities with the methods of GRAUERT and HIRONAKA.

I thank MICHAEL ARTIN and DAVID MUMFORD sincerely for these discussions, out of which this work materializes.

§1. Rationale Singularities

1.1 This section is a compilation of some of Artin's results on rational singularities (cf. [1, 2]). Let (X, x) be a 2-dimensional normal singularity, i.e. let (X, x) be the germ of a 2-dimensional reduced complex space and the local ring $\mathcal{O}_{X,x}$ at the point x is normal. (The terminology used in what follows regarding the germ (X, x) is hopefully understandable without further explanations and can be made precise by necessity; cf. also [13]).

Definition. *(X, x) is a rational singularity, if for a resolution $f : X' \to X$ the first direct image $R^1 f_* \mathcal{O}_{X'}$ of the structure sheaf $\mathcal{O}_{X'}$ of X' vanishes at x.*

One obtains a criterion for the rationality of (X, x) as follows. Let $f : X' \to X$ be a resolution of the singularity. Let the curves $C_1, \ldots, C_k$ be the irreducible components of $f^{-1}(x)$. A cycle $Z = r_1 C_1 + \cdots + r_k C_k$ is called positive, if not all r_i vanish and $r_i \geqq 0$ for $i = 1, \ldots, k$. The set of those positive cycles Z partially ordered by this definition, whose intersection numbers $Z \cdot C_i \leqq 0$ for $i = 1, \ldots, k$, contains a minimal element Z_0.

Definition. *Z_0 is called the fundamental cycle of the resolution f of the singularity (X, x).*

The virtual class $p(Z)$ for a cycle Z is defined as usual through the following formula, in which K is the canonical divisor of X',

$$p(Z) = \frac{1}{2}(Z \cdot Z + K \cdot Z) + 1.$$

Now let f be an arbitrary fixed resolution of (X, x) and Z_0 the fundamental cycle. Then, it holds ([2] Prop. 1, Theorem 3, Corollary 6) that:

Theorem 1.1 (Artin)**.** *The following statements are equivalent:*

i) *(X, x) is rational.*

ii) *$p(Z) \leqq 0$ for each positive cycle Z.*

iii) *$p(Z_0) = 0$ for the fundamental cycle Z_0.*

Theorem 1.2 (Artin). *Let (X, x) be a rational singularity, $\mathcal{O}_{X,x}$ its local ring, and $\mathfrak{m}$ its maximal ideal. Then, the following holds for the multiplicity $e(\mathcal{O}_{X,x})$ and the embedding dimension* $\operatorname{ebdim}(\mathcal{O}_{X,x}) = \dim_{\mathcal{O}_{X,x}/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2$, *resp.*

$$\begin{aligned} e(\mathcal{O}_{X,x}) &= -Z_0 \cdot Z_0 \\ \operatorname{ebdim}(\mathcal{O}_{X,x}) &= -Z_0 \cdot Z_0 + 1 \end{aligned}$$

It trivially results from Theorem 1.1 that

Lemma 1.3. *It holds for the system of exceptional curves $\{C_i\}$ of a resolution of a rational singularity that:*

i) All C_i are singularity-free and rational.

ii) $C_i \cap C_j \cap C_k = \varnothing$ for pairwise distinct i, j, k.

iii) $\{C_i\}$ is cycle-free.

Because of (iii), one can describe the negative definite intersection matrix $(C_i \cdot C_j)$ in an easy way, through a valued graph whose points correspond to the C_i and are valued with $+C_i \cdot C_i$, and whose edges connect pairs of vertices $\{C_i, C_j\}$ with $C_i \cdot C_j = 1$. Because of (iv), this graph is a tree. In this way, for example due to Theorem 1.2, the minimal resolutions of rational double points as valued graphs – up to the sign of the values – results in exactly the Dynkin diagrams of those simple Lie algebras, whose roots have the same length. ARTIN has listed the valued graphs belonging to $e = 3$ in [2].

Definition. *A regular resolution is a resolution of a 2-dimensional singularity with the properties from Lemma 1.3.*

1.2 This section handles the connection between rational singularities and near-factorial local rings.

Definition. *Let R be a zero-divisor-free commutative ring with unity. R is called factorial if every element of R different from zero, which is not a unit, is a product of prime elements. R is called near-factorial, if for each element of R different from zero, which is not a unit, a power x^n is a product of prime elements.*

It holds that:

Proposition 1.4. *Let R be a Krull ring, and $C(R)$ be its divisor class group. Then it holds that*

i) R is factorial, exactly if $C(R) = 0$.

ii) R is near-factorial, exactly if $C(R)$ is a torsion group.

Statement (i) is well-known (see e.g. [3] §3).

Statement (ii) is proven in [29] §1, Satz 1.

Now let R be the local ring of a 2-dimensional normal singularity (X, x). Furthermore, let $f : X' \to X$ be a resolution of singularities, $C = f^{-1}(x)$, and $C_1, \dots, C_k$ be the irreducible components of C. Let the resolution be chosen so that the C_i are singularity-free and intersect themselves transversely. From the usual short exact sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_{X'} \to \mathcal{O}^*_{X'} \to 0$$

and the corresponding long exact sequence of the image sheaves, MUMFORD deduced an exact sequence

$$0 \to H^1(C, \mathbb{Z}) \to \left(R^1 f_* \mathcal{O}_{X'}\right)_x \to \left(R^1 f_* \mathcal{O}^*_{X'}\right)_x \to H^2(C, \mathbb{Z}) \to 0. \tag{1}$$

Furthermore, there is a canonical surjection.

$$(R^1 f_* \mathcal{O}^*_{X'})_x \to C\left(\mathcal{O}_{X,x}\right) \to 0, \tag{2}$$

whose kernel consists of the group of cycles $\sum n_i C_i$. Let M be the boundary of a "good" neighborhood of x. (One such neighborhood can be constructed e.g. as in MUMFORD or with respect to a local embedding of (X, x) as the intersection of X with a small ball around x.) Let $H_1(M, \mathbb{Z})_0$ be the torsion subgroup of $H_1(M, \mathbb{Z})$. Then, MUMFORD obtains an exact sequence from 1 and 2.

$$0 \to H^1(C, \mathbb{Z}) \to \left(R^1 f_* \mathcal{O}_{X'}\right)_x \to C\left(\mathcal{O}_{X,x}\right) \to H_1(M, \mathbb{Z})_0 \to 0. \tag{3}$$

From 3 and 1.4 (ii) (cf. STORCH [29] §6, Satz 1):

Proposition 1.5. *A two-dimensional normal singularity (X, x) is rational, exactly if $\mathcal{O}_{X,x}$ is near-factorial.*

Because of Lemma 1.3, in the case of rational singularities, an easy-to-prove corollary of HIRZEBRUCH in [16] p. 04 is applicable towards the calculation of $H_1(M, \mathbb{Z})$, and one obtains for rational (X, x)

$$C\left(\mathcal{O}_{X,x}\right) \cong H_1(M, \mathbb{Z})$$
$$\operatorname{ord} C\left(\mathcal{O}_{X,x}\right) = \left|\det\left(C_i \cdot C_j\right)\right|.$$

Corollary 1.6. *A 2-dimensional analytic local ring $\mathcal{O}_{X,x}$ is factorial, exactly if (X, x) is rational and* $\det\left(C_i \cdot C_j\right) = \pm 1$.

Remark. *The statements of 1.5 and 1.6 applies allegedly for rings other than analytic local rings. Towards the proof of corresponding generalizations, one must however replace the trancendental methods with others. C.f. addition in the erratum.*

1.3 The following theorem about rational singularities and covers can be concluded, due to Proposition 1.5, from a corresponding theorem of STORCH about near-factorial local rings. ([29], § 3, Satz 2). I declare nevertheless another proof, because this goes back directly to the definition of rational singularities, does not involve the transcendental methods used in the proof of Proposition 1.5, and can be generalized beyond the analytic case. This proof was shared to me by MUMFORD.

Proposition 1.7. *Let (X,x) and (Y,y) be normal 2-dimensional singularities, $u : (X,x) \to (Y,y)$ a cover, and (X,x) rational. Then, (Y,y) is rational.*

Beweis. Without loss of generality, it is assumed that X *resp.* Y have singularities at most in x *resp.* y. Let $g : Y' \to Y$ be a resolution of singularities for Y. $Y.m = g^{-1} \circ u$ is meromorphic; let G_m be the graph, $p' : G_m \to X$, and $p'' : G_m \to Y'$ be the projections. Let $n : X' \to G_m$ be a resolution of singularities for G_m. With $v = p' \circ n$ and $f = p'' \circ n$ as well as $h = gf$, one has a commutative diagram

$$\begin{tikzcd} X' \arrow[r, "f"] \arrow[d, "v"'] \arrow[dr, "h"] & Y' \arrow[d, "g"] \\ X \arrow[r, "u"'] & Y \end{tikzcd}$$

One has (EGA III 12.2.4 or Tohuku) two spectral sequences

i) $E_2^{pq} = R^p g_* \left(R^q f_* \mathcal{O}_{X'}\right) \Rightarrow R^* h_* \mathcal{O}_{X'}$,

ii) $E_2'^{pq} = R^p u_* \left(R^q v_* \mathcal{O}_{X'}\right) \Rightarrow R^0 h_* \mathcal{O}_{X'}$.

Because (X,x) is a rational singularty and $v : X' \to X$ is a resolution of singularities of X, [we have] $R^q v_* \mathcal{O}_{X'} = 0$ for $q > 0$ and $R^0 v_* \mathcal{O}_{X'} = \mathcal{O}_X$. Because u is discrete and proper, [we have] $R^p u_* F = 0$ for $p > 0$ and each coherant sheaf F. Thus, $E_2'^{pq} = 0$ for $(p,q) \neq (0,0)$ and $E_2'^{00} = u_* \mathcal{O}_X$. Thus, $R^k h_* \mathcal{O}_{X'} = 0$ for $k > 0$. Since f and g have at most 1-dimensional fibers, $E_2^{pq} = 0$ for $p > 1$ or $q > 1$. Thus, $R^1 g_* \left(f_* \mathcal{O}_X\right) = E_2^{1,0} = E_\infty^{1,0} = 0$ because $R^1 h_* \mathcal{O}_{X'} = 0$. Then, it also holds, as we will show, that $R^1 g_* \mathcal{O}_{Y'} = 0$, because $\mathcal{O}_{Y'}$ is a direct summand of $f_* \mathcal{O}_{X'}$. Proof of the latter: let $X' \xrightarrow{a} Y'' \xrightarrow{b} Y'$ be the Stein factorization of f. Y'' is normal, a connected, and b a cover. It holds that $f_* \mathcal{O}_{X'} = b_* a_* \mathcal{O}_{X'} = b_* \mathcal{O}_{Y''}$. One has a natural injection $i : \mathcal{O}_{Y'} \to b_* \mathcal{O}_{Y''}$, but also the ***trace mapping (Spurabbildung) $t : b_* \mathcal{O}_{Y''} \to \mathcal{O}_{Y'}$. It holds that $t \circ i = c.\mathrm{Id}$ ($c =$ degree of the cover), and therefore $\mathcal{O}_{Y'}$ is a direct summand. (Definition of t: for y in Y', over which b is not branched, we have $\left(b_* \mathcal{O}_{Y''}\right)_y = \mathcal{O}^c_{Y',y}$, and $t\left(f_1, \ldots, f_c\right) = f_1 + \cdots + f_c$.) It is steadily continued in the branching set ***??. $\square$

§2. Singularities of Quotient Spaces

2.1 This section is a compilation of the results used in what follows, of CARTAN and PRILL about quotients of complex spaces (cf. [7,22]). Let X be a normal complex space, G a proper discontinuous group of automorphisms of X. The topological quotient space X/G is provided as follows with a structure sheaf of functional germs $\mathcal{O}_{X/G}$: Let $p : X \to X/G$ be the map of residual classes. Then, it holds for each open set U of X/G that

$$\mathcal{O}_{X/G}(U) = \left\{ f \mid f \circ p \in \mathcal{O}_X \left(p^{-1}(U) \right) \right\}.$$

Theorem 2.1 (Cartan)**.** *i) $(X/G, \mathcal{O}_{X/G})$ is a normal complex space.*

ii) The map $X \to X/G$ is holomorphic, surjective, discrete, and proper for finite G, so it is therefore an analytic branched cover.

Definition. *A quotient singularity is a singularity which is isomorphic to a singularity of a quotient X/G of a complex manifold X by a proper discontinuous group G.*

The following statement about quotient singularities is well-known (CARTAN [7], p. 97).

Lemma 2.2. *Each quotient singularity is isomorphic to a singularity $(\mathbb{C}^n/G, 0)$, where G is a finite subgroup of $GL(n, \mathbb{C})$, and 0 is the point corresponding to the origin of $\mathbb{C}^n$.*

Beweis. The singularity of X/G where 0 is the corresponding point is isomorphic to the corresponding singularity of X/G_0, where G_0 is the (finite) isotropy group of 0. Let $(z_1, \ldots, z_n)$ be complex coordinates in a neighborhood of $0 = (0, \ldots, 0)$. One introduces in an appropriate neighborhood of 0 new coordinates z' through

$$z' = \sum_{g \in G_0} g'^{-1} g z$$

where $g' = (\partial g / \partial z)_0$. Then, G_0 operates linearly with respect to these coordinates. □

PRILL has completely classified the quotient singularities $(\mathbb{C}^n/G, 0)$ in [22]:

Definition. *A subgroup G of $GL(n, \mathbb{C})$ is called small, exactly if no $g \in G$ has the number* 1 *as an eigenvalue of multiplicity $n - 1$.*

Theorem 2.3 (PRILL)**.** *i) Each quotient singularity is isomorphic to a singularity $(\mathbb{C}^n/G, 0)$, where G is a finite small subgroup of $GL(n, \mathbb{C})$.*

ii) Let G and G' be small subgroups of $GL(n, \mathbb{C})$. Then, the singularities $(\mathbb{C}^n/G, 0)$ and $(\mathbb{C}^n/G', 0)$, exactly if G and G' are conjugate.

One compares this also with the work of GOTTSCHLING [11]. If $G \subset GL(n, \mathbb{C})$ is an arbitrary finite [sub]group, and H is the normal subgroup generated by the elements with 1 as eigenvalue of multiplicity $n - 1$, then $\mathbb{C}^n/H$ is singularity-free and the "reduced" group $\bar{G} = G/H$ operates equivalently to a small [sub]group on $\mathbb{C}^n/H$.

2.2 The proof presented in this work for the uniqueness of the icosahedral singularity is a transcendental proof, because it uses the topology of the singularity: more precisely, the local fundamental group. In this section, some well-known results in this regard will be summarized.

Let X be a complex space, $x \in X$, and X be irreducible at x. A neighborhood U of x in X is called after PRILL a *good neighborhood*, if there is a neighborhood basis $\{U_i\}$ of x, such that each $U_i - x$ is a deformation retract of $U - x$. For all good neighborhoods U, $U - x$ has the same homotopy type, and one can thus in particular define the local fundamental group $\pi_{X,x}$ of X at x. In order to make the definition also formally independent from U, one can define

$$\pi_{X,x} = \underline{\lim}\, \pi_1(U - x),$$

whereby U goes through the system of neighborhoods of x, or what amounts to the same thing, the cofinal subsystem of the good neighborhoods. Thereby is the above definition to be interpreted as in [14], Exposé XIII and Commentaires à l'Exposé XIII.

The fundamental fact for 2-dimensional singularities is the following result from MUMFORD.

Theorem 2.4 (MUMFORD). *Let (X, x) be a 2-dimensional normal singularity and $\pi_{X,x} = 1$. Then, $\mathcal{O}_{X,x}$ is regular.*

Corollary 2.5 (MUMFORD). *Let (X, x) be a 2-dimensional normal singularity and X a topological manifold at x. Then, X is not singular at x.*

Remark. *From the examples, which I have described in [5], it comes out that there is no statements corresponding to this corollary for higher dimensions. Therefore, some of the following arguments for 2-dimensional singularities do not let themselves transfer to higher dimensions.*

Lemma 2.6. *Let $f : (X, x) \to (Y, y)$ be a cover of normal singularities and $\pi_{X,x}$ be finite. Then, $\pi_{Y,y}$ is also finite.*

Beweis. Without loss of generality, let $f^{-1}(y) = x$. Let V_1 be a good neighborhood of y; U_1 a good neighborhood of x with $f(U_1) \subset V_1$; furthermore V_2 a good neighborhood of y with $V_2 \subset V_1$ and $f^{-1}(V_2) \subset U_1$; finally, let $U_2 = f^{-1}(V_2)$. Let $V_i^- = V_i - \{y\}$ and $U_i^- = U_i - \{x\}$. Let $\widetilde{V}_1 \to V_1^-$ be the universal cover of V_1^- and $\widetilde{U}_1 \to U_1^-$ the universal cover of U_1^-. $\widetilde{U}_1$ and $\widetilde{V}_1$ are endowed with a complex structure in a canonical way, such that the covering map is locally biholomorphic. Therefore, the fiber products

$$\widetilde{U}_2 = U_2^- \times_{U_1^-} \widetilde{U}_1 \quad \text{and} \quad \widetilde{V}_2 = V_2^- \times_{V_1^-} \widetilde{V}_1 \quad \text{and}\ \widetilde{\widetilde{U}}_i = \widetilde{U}_i \times_{V_i^-} \widetilde{V}_i$$

also reduced normal complex spaces. Because $\widetilde{V}_1 \to V_1^-$ is an unbranched topological cover, so is $\widetilde{\widetilde{U}}_1 \to \widetilde{U}_1$ an unbranched topological cover and has, due to the easy connection with $\widetilde{U}_1$, a section. Then, $\tilde{\tilde{U}}_2 \to \widetilde{U}_2$ also has a section, and this is a holomorphic map $s : \widetilde{U}_2 \to \tilde{\tilde{U}}_2$. Through composition of s with $\tilde{\tilde{U}}_2 \to \widetilde{V}_2$, one obtains a commutative diagram of holomorphic maps.

$$\begin{array}{ccc} \widetilde{U}_2 & \longrightarrow & \widetilde{V}_2 \\ \downarrow & & \downarrow \\ U_2^- & \longrightarrow & V_2^- \end{array}$$

Claim: $\widetilde{U}_2 \to \widetilde{V}_2$ is surjective. Proof: The map is proper, discrete, holomorphic, so the image is a (closed) analytic subset of $\widetilde{V}_2$ with the same dimension as $\widetilde{V}_2$, therefore a connected component of $\widetilde{V}_2$. However, $\widetilde{V}_2$ is connected, because $\widetilde{V}_1$ is connected and the inclusion of good neighborhoods $V_2^- \to V_1^-$ induces an isomorphism of the fundamental groups.

Let $O_x(f)$ be the degree of f at x, and $O\left(\pi_{X,x}\right)$ resp. $O\left(\pi_{X,y}\right)$ the orders of the local fundamental groups. Then, it follows, due to the surjectivity of $\widetilde{U}_2 \to \widetilde{V}_2$, from the above diagram that

$$O\left(\pi_{Y,y}\right) \leqq O_x(f) O\left(\pi_{X,x}\right)$$

and therewith has the finiteness of $\pi_{Y,y}$ been proven. □

Let (X, x) be a 2-dimensional normal singularity, with a regular resolution with system of exceptional curves $C_1, \ldots, C_k$. Let $s_{ij} = C_i \cdot C_j$. Then, one can calculate $\pi_{X,x}$ alone from the intersection matrix (s_{ij}), since it holds (cf. MUMFORD [21], p.12; HIRZEBRUCH [16]) that:

Lemma 2.7. *Under the above requirements, $\pi_{X,x}$ is generated with k elements $e_1, \ldots, e_k$ with the following relations:*

$$e_i e_j^{s_{ij}} = e_j^{s_{ij}} e_i$$
$$e_1^{s_{i1}} e_2^{s_{i2}} \ldots e_k^{s_{ik}} = 1.$$

2.3 From Lemma 2.6 it follows in particular, that a singularity, which admits a singularity-free cover, has a finite local fundamental group. The converse thereof does not hold in general (cf. however PRILL [22], Proposition 5). For 2-dimensional singularities, however, the following holds.

Proposition 2.8. *Let (Y, y) be a -dimensional normal singularity. Then, the following statements are equivalent:*

i) (Y, y) is a quotient singularity.

ii) There exists a cover $(X, x) \to (Y, y)$ with regular $\mathcal{O}_{X,x}$.

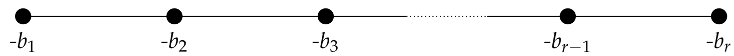
iii) $\pi_{Y,y}$ is finite.

Beweis. (i) implies (ii) due to Theorem 2.1 and (ii) implies (iii) due to Lemma 2.6. To be shown is, that (iii) implies statement (i). Let V be a sufficiently small good neighborhood of y in Y, $V' = V - \{y\}$, and $U' \to V'$ the universal cover, which has finite leaves because of the finiteness of $\pi_{Y,y}$. One can (e.g. following FOX [9]) continue this cover to a branched topological map $U \to V$ through adding a point x in a definite way. Following the fundamental result of the work of GRAUERT and REMMERT about complex spaces [12], one can endow U with a normal complex Structure, so that $U \to V$ is an analytic branched cover. From the easy connection with U' follows $\pi_{U,x} = 1$, and therefore $\mathcal{O}_{U,x}$ is regular due to Satz 2.4 of MUMFORD. $\pi_{Y,y}$ operates through deck transformations holomorphically on U' and also on U with fixed point x and quotient $U/\pi_{Y,y} = V$. Thus, (V, y) is a quotient singularity. □

2.4 a) Theorem 2.3 of PRILL leads back to the classification of the quotient singularities described in Proposition 2.8 via the enumeration of the conjugacy classes of small subgroups of $GL(2, \mathbb{C})$. In order to obtain further information about the quotient singularities, one can e.g. calculate their local rings using invariant theory. This is for the subgroups of $SL(2, \mathbb{C})$ that were carried out e.g. from KLEIN [19], Kap.II, § 9-13 and DUVAL [22], p. 94-112. Another method consists of resolving the quotient singularities. One obtains from the resolution, following Theorems 1.2 and 1.7, the multiplicities and embedding dimension of the local ring. It shows that the quotient singularities are classified also by the intersection matrix of their resolutions. This section handles the connection between the two mentioned classifications.

b) The intersection matrix can be described through the corresponding valued trees. It will turn out, that for the quotient singularities the trees are all straight-shaped or star-shaped with three branches. These sorts of valued trees can be described in the following manner.

Definition. *Let n and q be coprime whole numbers $0 < q < n$. Then, let $\langle n, q\rangle$ be the valued tree*



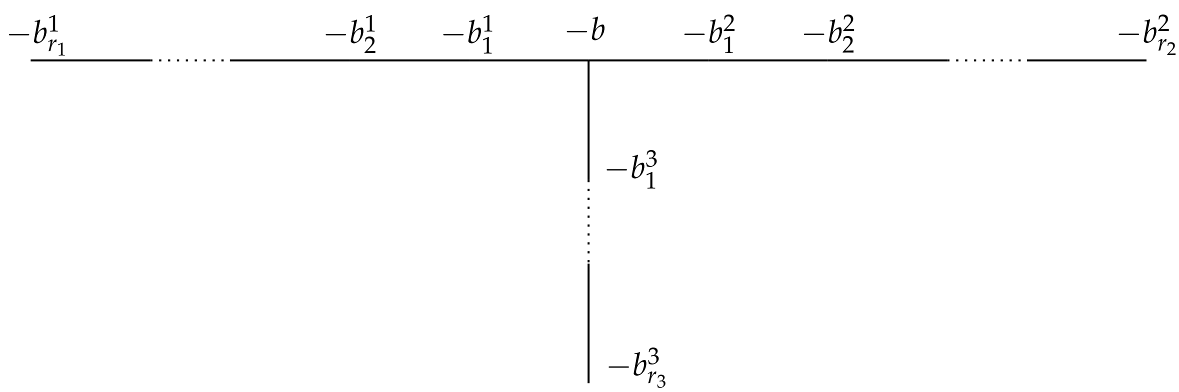
where the b_i are whole numbers, which are unambiguously characterized through the following relations:

$$b_i \geqq 2 \text{ for } i = 1, \dots, r$$

$$\frac{n}{q} = b_1 - \cfrac{1}{b_2 - \cfrac{1}{b_3 \ddots \cfrac{1}{b_{r-1} - \frac{1}{b_r}}}} \tag{4}$$

Following [23], p. 61, $\langle n, q \rangle$ and $\langle n', q' \rangle$ are the same valued trees exactly if $n = n'$ and $q = q'$ or $qq' \equiv 1(n)$.

Definition. *Let $\langle b; n_1, q_1; n_2, q_2; n_3, q_3 \rangle$ be the valued tree*



where the b, b^i_k are whole numbers with $b, b^i_k \geqq 2$ and the whole numbers n_i, q_i with $(n_i, q_i) = 1$ and $0 < q_i < n_i$ are given through the following chained fractions.

$$\frac{n_i}{q_i} = b^i_1 - \cfrac{1}{b^i_2 - \cfrac{1}{b^i_3 \ddots \cfrac{1}{b^i_{r-1} - \frac{1}{b^i_r}}}}.$$

c) It is clear, that one can describe the subgroups of $GL = GL(2,\mathbb{C})$ in the following manner: let $SL \subset GL$ be the special linear group, $ZL \subset GL$ the centre and $\psi : ZL \times SL \to GL$ the multiplication. Let H_1 resp. H_2 be subgroups of ZL resp. SL and N_i be normal subgroups of H_i, for which H_1/N_1 and H_2/N_2 are isomorphic; let $\varphi : H_2/N_2 \to H_1/N_1$ be an isomorphism. Let $\bar{h}_i$ denote the residue class of h_i in H_i/N_i and

$$H_1 \times_\varphi H_2 = \left\{(h_1, h_2) \in H_1 \times H_2 \mid \bar{h}_1 = \varphi\left(\bar{h}_2\right)\right\}$$

the fibered product. Finally, let

$$(H_1, N_1; H_2, N_2)_\varphi = \psi\left(H_1 \times_\varphi H_2\right).$$

Each finite subgroup of GL is of the form $(H_1, N_1; H_2, N_2)_\varphi$, and it is not difficult to notice that the conjugacy class depends on $H_1, N_1; H_2, N_2)_\varphi$. In fact, it turns out that in almost all cases it does not depend on ϕ. In these cases, the φ is omitted therefrom, and $(H_1, N_1; H_2, N_2)$ denotes a fixed representation of the corresponding conjugacy class.

The only case, in which φ plays a role, is that where H_1 and H_2 are cyclic. In this case, however, the following description of the group is more practical. One sees easily, that each non-trivial small [sub]group $(H_1, N_1; H_2, N_2)_\varphi$ with cyclic H_i is conjugate to one of the defined-as-follows cyclic groups $\mathsf{C}_{n,q}$:

$$\mathsf{C}_{n,q} = \left\{\begin{pmatrix} e^{2\pi i/n} & 0 \\ 0 & e^{2\pi i q/n} \end{pmatrix}\right\} \quad 0 < q < n, (n,q) = 1$$

$\mathsf{C}_{n,q}$ and $\mathsf{C}_{n',q'}$ are naturally conjugate exactly if $n' = n$ and $q = q'$ or $qq' \equiv 1(n)$.

The Groups H_i are denoted in the following manner:

Z_k	the cyclic group of order k in ZL,
C_k	the cyclic group of order k in SL,
D_k	the binary dihedral group of order $4k$,
T	the binary tetrahedral group,
O	the binary octahedral group,
I	the binary icosahedral group.

The subgroups of SL in this list are naturally determined only up to conjugation. Let one representation be fixed for each, and indeed so, that one has normal subgroups $\mathsf{C}_{2n} \triangleleft \mathsf{D}_n$ and $\mathsf{D}_2 \triangleleft \mathsf{T}$.

d) Following these preparations, the 2-dimensional singularities can be classified in the following manner up to an analytic isomorphism, where each of the regular singularties resp. the trivial group resp. the empty graph are omitted.

Theorem 2.9. *i) The 2-dimensional quotient singularities are classified through the conjugacy classes of small subgruops of $GL(2,\mathbb{C})$. Each small subgroup of $GL(2,\mathbb{C})$ is conjugate to one of following groups:*

$$\begin{array}{ll}
\mathsf{C}_{n,q} \quad 0<q<n, & (n,q)=1 \\
(\mathsf{Z}_{2m},\mathsf{Z}_{2m};\mathsf{D}_n,\mathsf{D}_n) & (m,2)=1,(m,n)=1 \\
(\mathsf{Z}_{4m},\mathsf{Z}_{2m};\mathsf{D}_n,\mathsf{C}_{2n}) & (m,2)=2,(m,n)=1 \\
(\mathsf{Z}_{2m},\mathsf{Z}_{2m};\mathsf{T},\ \mathsf{T}) & (m,6)=1 \\
(\mathsf{Z}_{6m},\mathsf{Z}_{2m};\mathsf{T},\mathsf{D}_2) & (m,6)=3 \\
(\mathsf{Z}_{2m},\mathsf{Z}_{2m};\mathsf{O},\mathsf{O}) & (m,6)=1 \\
(\mathsf{Z}_{2m},\mathsf{Z}_{2m};\mathsf{I},\mathsf{I}) & (m,30)=1.
\end{array}$$

ii) $\mathsf{C}_{n,q}$ and $\mathsf{C}_{n',q'}$ are conjugate exactly if $n=n'$ and $q=q'$ or $qq'\equiv 1(n)$. The remaining groups are not conjugate.

Beweis. This theorem is a immediate consequence from Theorem 2.3 of PRILL, of the definition of small [sub]groups and of the well-known classification of the finite subgroups of $GL(2,\mathbb{C})$ (cf. e.g. [8] p. 57). □

Theorem 2.10. *i) The 2-dimensional quotient singularities are exactly the 2-dimensional normal singularities, which have a regular resolution with one of the following valued graphs: $\langle n,q\rangle$ with $0<q<n$ and $(n,q)=1$, $\langle b;n_1,q_1;n_2,q_2;n_3,q_3\rangle$ with $b\geqq 2$, $0<q_i<n_i$ and $(n_i,q_i)=1$, where (n_1,n_2,n_3) is a platonic triple $(2,2,n)$, $(2,3,3)$, $(2,3,4)$, $(2,3,5)$.*

ii) These valued graphs of the minimal resolution classify the quotient singularities up to an analytic isomorphism.

Beweis. Theorem 2.10 is an immediate consequence of Lemma 2.7 and Proposition 2.8 and the following theorem. □

Theorem 2.11. *The following table contains in column 1 the small subgroups G of $GL(2,\mathbb{C})$, in column 2 the valued graphs Γ_G of the minimal resolution of the quotient singularity $(\mathbb{C}^2/G,0)$, in column 3 the multiplicity e_G of the local ring $\mathcal{O}_{\mathbb{C}^2/G,0}$, and in column 4 the divisor class groups $D_G=G/G'$ of $\mathcal{O}_{\mathbb{C}^2/G,0}$.*

G	Γ_G		e_G	D_G
$\mathsf{C}_{n,q}$	$\langle n, q\rangle$	$0 < q < n, (n,q) = 1$	$2 + \Sigma(b_i - 2)$	Z_n
$(\mathsf{Z}_{2m}, \mathsf{Z}_{2m}; \mathsf{D}_n, \mathsf{D}_n)$	$\langle b; 2, 1; 2, 1; n, q\rangle$	$m = (b-1)\ n - q$, $2 \nmid m$	$2 + \Sigma(b_i - 2)$	$\mathsf{Z}_{2m} \times \mathsf{Z}_2$ $2 \mid n$
$(\mathsf{Z}_{4m}, \mathsf{Z}_{2m}; \mathsf{D}_n, \mathsf{C}_{2n})$	$\langle b; 2, 1; 2, 1; n, q\rangle$	$m = (b-1)n - q$, $2 \mid m$		Z_{4m} $2 \nmid n$
$(\mathsf{Z}_{2m}, \mathsf{Z}_{2m}; \mathsf{T}, \mathsf{T})$	$\langle b; 2, 1; 3, 2; 3, 2\rangle$	$m = 6(b-2) + 1$	b	Z_{3m}
$(\mathsf{Z}_{2m}, \mathsf{Z}_{2m}; \mathsf{T}, \mathsf{T})$	$\langle b; 2, 1; 3, 1; 3, 1\rangle$	$m = 6(b-2) + 5$	$b+2$	Z_{3m}
$(\mathsf{Z}_{6m}, \mathsf{Z}_{2m}; \mathsf{T}, \mathsf{D}_2)$	$\langle b; 2, 1; 3, 1; 3, 2\rangle$	$m = 6(b-2) + 3$	$b+1$	Z_{3m}
$(\mathsf{Z}_{2m}, \mathsf{Z}_{2m}; \mathsf{O}, \mathsf{O})$	$\langle b; 2, 1; 3, 2; 4, 3\rangle$	$m = 12(b-2) + 1$	b	Z_{2m}
$(\mathsf{Z}_{2m}, \mathsf{Z}_{2m}; \mathsf{O}, \mathsf{O})$	$\langle b; 2, 1; 3, 1; 4, 3\rangle$	$m = 12(b-2) + 5$	$b+1$	Z_{2m}
$(\mathsf{Z}_{2m}, \mathsf{Z}_{2m}; \mathsf{O}, \mathsf{O})$	$\langle b; 2, 1; 3, 2; 4, 1\rangle$	$m = 12(b-2) + 7$	$b+2$	Z_{2m}
$(\mathsf{Z}_{2m}, \mathsf{Z}_{2m}; \mathsf{O}, \mathsf{O})$	$\langle b; 2, 1; 3, 1; 4, 1\rangle$	$m = 12(b-2) + 11$	$b+3$	Z_{2m}
$(\mathsf{Z}_{2m}, \mathsf{Z}_{2m}; \mathsf{I}, \mathsf{I})$	$\langle b; 2, 1; 3, 2; 5, 4\rangle$	$m = 30(b-2) + 1$	b	Z_m
$(\mathsf{Z}_{2m}, \mathsf{Z}_{2m}; \mathsf{I}, \mathsf{I})$	$\langle b; 2, 1; 3, 2; 5, 3\rangle$	$m = 30(b-2) + 7$	$b+1$	Z_m
$(\mathsf{Z}_{2m}, \mathsf{Z}_{2m}; \mathsf{I}, \mathsf{I})$	$\langle b; 2, 1; 3, 1; 5, 4\rangle$	$m = 30(b-2) + 11$	$b+1$	Z_m
$(\mathsf{Z}_{2m}, \mathsf{Z}_{2m}; \mathsf{I}, \mathsf{I})$	$\langle b; 2, 1; 3, 2; 5, 2\rangle$	$m = 30(b-2) + 13$	$b+1$	Z_m
$(\mathsf{Z}_{2m}, \mathsf{Z}_{2m}; \mathsf{I}, \mathsf{I})$	$\langle b; 2, 1; 3, 1; 5, 3\rangle$	$m = 30(b-2) + 17$	$b+2$	Z_m
$(\mathsf{Z}_{2m}, \mathsf{Z}_{2m}; \mathsf{I}, \mathsf{I})$	$\langle b; 2, 1; 3, 2; 5, 1\rangle$	$m = 30(b-2) + 19$	$b+3$	Z_m
$(\mathsf{Z}_{2m}, \mathsf{Z}_{2m}; \mathsf{I}, \mathsf{I})$	$\langle b; 2, 1; 3, 1; 5, 2\rangle$	$m = 30(b-2) + 23$	$b+2$	Z_m
$(\mathsf{Z}_{2m}, \mathsf{Z}_{2m}; \mathsf{I}, \mathsf{I})$	$\langle b; 2, 1; 3, 1; 5, 1\rangle$	$m = 30(b-2) + 29$	$b+4$	Z_m

Beweis. (a) First, the calculation of Γ_G : In the case that G is cyclic, i.e. $\mathsf{C}_{n,q}$, Γ_G is already calculated in [15] using the Hirzebruch-Jung algorithm, i.e. using the modified continued fraction expansion (4) for n/q. The general case can be reduced to the cyclic case as follows. Let X be the surface resulting from $\mathbb{C}^2$ by σ-process in 0 and C be the exceptional curve of X. The group G operates on X and X/G is a modification of $\mathbb{C}^2/G$ in which the singular point is replaced by the rational curve C' corresponding to C. According to 2.3, the singular points of X/G correspond to the points $x \in C$ for which the reduced isotropy group $\bar{G}_x$ is not trivial. These are, if $G = (H_1, N_1; H_2, N_2)_\varphi$, at most the points for which the image of H_2 in $PGL(1, \mathbb{C})$ has nontrivial isotropy groups as transformation group of P_1. This is known to be the case for exactly 2 point groups if H_2 is cyclic, and for 3 point groups in all other cases. The isotropy groups of H_2 are cyclic, the G_x is therefore equivalent to certain $(\mathsf{Z}_{2mr}, \mathsf{Z}_{2m}; \mathsf{C}_{2nr}, \mathsf{C}_{2n})_\varphi$ and the $\bar{G}_x$ is therefore equivalent to certain $\mathsf{C}_{n,q}$, so that one can apply the result of HIRZEBRUCH. I will skip the calculation of the n, q. Due to Lemma 1.3 it is clear that at the minimal resolution $\widetilde{X}$ of X/G the curve C_0 corresponding to C' is also free of singularities and rational and intersects the other exceptional curves transversally. One easily sees that C_0 actually intersects as claimed in 2.11. Thus, in the noncyclic case, to determine $\Gamma_G = \langle b; n_1, q_1; n_2, q_2;\ n_3, q_3\rangle$ only $b = -C_0 \cdot C_0$ remains to be calculated. Let $G = (Z_{2mr}, Z_{2m};\ H_2, N_2)$ and h be the order of the image of G in $PGL(1, \mathbb{C})$. From the fact that the divisor of a holomorphic function on $\widetilde{X}$ must intersect every exceptional curve with multiplicity 0, one easily obtains the following formula for b

$$b = \frac{q_1}{n_1} + \frac{q_2}{n_2} + \frac{q_3}{n_3} + \frac{2m}{h}.$$

(b) The calculation of e_G follows from that of Γ_G because of Theorem 1.2 and Proposition 1.7. For, if $Z_0 = \sum r_i C_i$ is the fundamental cycle of a rational singularity and $b_i = -C_i \cdot C_i$, then from 1.2 and 1.3 follows

$$e = 2 + \sum r_i (b_i - 2).$$

One easily sees, that for the fundamental cycle of all Γ_G from 2.11 with the valuations $-b_i$ from $b_i > 2$ it follows that $r_i = 1$. Therefore,

$$e_G = 2 + \sum (b_i - 2)$$

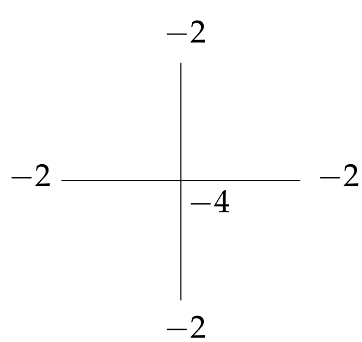
and the values of the table result from that.

(c) The divisor class group D_G of the rational singularity $(X, x) = \left(\mathbb{C}^2/G, 0\right)$ is, according to the remarks following Proposition 1.5, isomorphic to $H_1\left(S^3/G, \mathbb{Z}\right)$, i.e. to the quotient of $G = \pi_{X,x}$ with respect to its commutator [sub]group G'. This allows one to calculate $D_G = G/G'$ without difficulty from G. $\square$

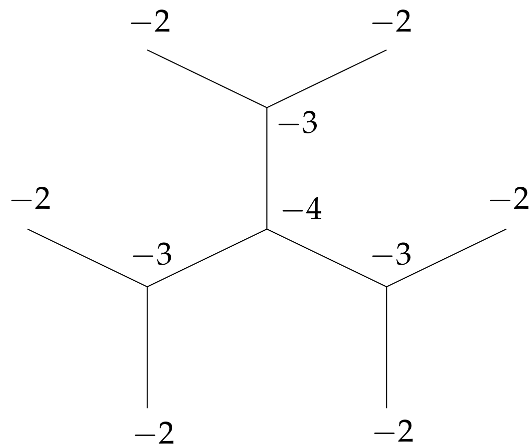
2.5 As a trivial consequence from Theorem 2.10, one obtains the following generalization of Satz 1 in [4]. In deviation from the usual word usage, it is established that:

Definition. *Let (X, x) be a 2-dimensional normal singularity with a minimal regular resolution and associated valued graph Γ. (X, x) is rigid, if (X, x) is the unique singularity, up to isomorphism, with a regular resolution with valued graph Γ.*

Of course, (X, x) cannot be rigid if Γ has a point that is a vertex of more than 3 1-simplices. In terms of examples,



belongs to non-rigid rational singularities. However, even if (X,x) is rational and every point of Γ lies on at most 3 1-simplices, need (X,x) not be rigid. One example for such a Γ is, following a letter message from TJURINA and ARTIN:



It is therefore that the following trivial consequence from Theorem 2.10 all the more remarkable.

Corollary 2.12. *The 2-dimensional quotient singularities are rigid.*

This corollary is of course the reason for the the definition of rigidity given above. It would have been more natural and weaker to require, that (X,x) is determined through the first infinitesimal neighborhood of the system of exceptional curves. —- I would still like to note, that TJURINA is said to have shown: All rational triple points are rigid.

2.6 This section elucidates the connection between the analytic classification of the quotient singularities on one hand, and the topological investigations of SEIFERT and THRELFALL about the discontinuity domains of finite ***bewegungsgruppen of the three-dimensional sphere and about Seifert fiber spaces as in V.RANDOWS results about tree manifolds on the other hand.

(a) Every quotient singularity ($\mathbb{C}^2/G, 0$) has the 3-dimensional closed orientable manifold $M_G = S^3/G$ as boundary of a good neighborhood, where S^3 is the standard 3-sphere in $\mathbb{C}^2$ considered as a G-subspace. In [28], Ch. III, it is shown that the totality of the neighborhood boundaries M_G obtained in this way coincides with the set of all manifolds S^3/H, where H is an arbitrary finite subgroup of $SO(4)$. If H is fixed point free on S^3, then S^3/H is called a spherical space form following HOPF. The homeomorphism problem for these manifolds is completely solved: If G is cyclic, then M_G is a lens space, more precisely: $M_{C_{n,q}} = L(n,q)$, and according to BRODY [6] goes: $L(n,q)$ and $L\,(n',q')$ are homeomorphic exactly if $n = n'$ and $q \equiv \pm q'(n)$ or $qq' \equiv \pm 1(n)$. For non-cyclic small subgroups G of $GL(2,\mathbb{C})$ the classifications agree up to conjugation on the one hand and up to abstract isomorphism on the other hand, as can easily be shown through consideration of G/G' and $G/$ center (G) (cf. [28], §6). Hence, for non-cyclic small G_1, G_2, the manifold M_{G_1} is homeomorphic to M_{G_2} exactly if G_1 is conjugate to G_2. For a two-dimensional normal singularity (X, x) with non-cyclic finite $\pi_{X,x}$, the analytic structure is thus uniquely determined through the topological, up to isomorphism.

(b) Let U be the full torus

$$U = \left\{(z_1, z_2) \in \mathbb{C}^2 || z_1 |= 1, |z_2| \leqq 1\right\}$$

and V the circular disk

$$V = \left\{(z_1, z_2) \in \mathbb{C}^2 || z_1 |= 0, |z_2| \leqq 1\right\}.$$

U is oriented, and indeed, if $z_1 = e^{2\pi i\varphi}$ and $z_2 = x + iy$, through the sequence of coordinates (φ, x, y). Let $f : U \to V$ be the projection, that is $f(z_1, z_2) = z_2$. The groups $\mathsf{C}_{n,q}$ defined in 2.4 operates on U and V as subspaces of $\mathbb{C}^2$. Let $f_{n,q} : U/\mathsf{C}_{n,q} \to V/\mathsf{C}_{n,q}$ be the map of quotient spaces induced by f, and $U/\mathsf{C}_{n,q}$ be provided with the orientation induced from U.

Definition. *A Seifert fiber space is a map $f : X \to Y$ of a three-dimensional closed manifold X out of a 2-dimensional manifold Y with 1-spheres as fibers, which over suitable neighborhoods of the point of Y is either locally trivial or of the same type as an $f_{n,q}$. (For generalizations cf. [18].)*

The Seifert fiber spaces with an orientation for X and orientable Y are classified in [27] through a system of invariants

$$(O, o; p \mid \beta; \alpha_1, \beta_1; \ldots; \alpha_n, \beta_n).$$

Here, O resp. o refer to the orientability of X resp. Y, and p is the genus of Y. The whole number β is the closure invariant (for the definition cf. [27], p.181), and the coprime whole numbers α_i, β_i with $0 < \beta_i < \alpha_i$ establish the oriented type f_{α_i,β_i} with the exceptional fibers. With reversal of the orientation,

$$(O, o; p \mid \beta; \alpha_1, \beta_1; \ldots, \alpha_r, \beta_r)$$

changes into

$$(0, a; p \mid -\beta - r; \alpha_1, \alpha_1 - \beta_1; \ldots; \alpha_r, \alpha_r - \beta_r).$$

Now, let G be a finite small subgroup of $GL(2, \mathbb{C})$. The group G operates on S^3 and P_1, and thereby one obtains from the Hopf fibration $S^3 \to P_1$ a Seifert fiber space

$$S^3/G \to P_1/G.$$

$M_G = S^3/G$ is oriented as the neighborhood boundary of a complex singularity (for the orientation of the boundary of an oriented manifold see the textbook of topology by SEIFERT-THRELFALL), and thus M_G is an oriented Seifert fiber space. The Seifert invariants of M_G have been already calculated in [28]. Through comparison with Theorem 2.11, one obtains:

Lemma 2.13. *If the singularity $(\mathbb{C}^2/G, 0)$ has a resolution with the valued graph $\Gamma_G = \langle b; n_1, q_1; n_2, q_2; n_3, q_3 \rangle$, then its neighborhood boundary M_G as a Seifert fiber space has the invariant*

$$(O, o; 0 \mid -b; n_1, q_1; n_2, q_2; n_3, q_3)\,.$$

In [27] all $(O, o; 0 \mid \beta; \alpha_1, \beta_1; \ldots; \alpha_r, \beta_r)$ with finite fundamental group were enumerated. By comparison with the fiber invariants of M_G determined in [28] or in 2.11 and 2.13, the main theorem of [28] follows: The neighborhood boundaries of the 2-dimensional quotient singularities are precisely the total spaces of the orientable Seifert fibrations over S^2 with finite fundamental group.

(c) If a 2-dimensional normal singularity has a regular resolution with valued tree Γ, then a suitable neighborhood boundary M is the tree manifold corresponding to Γ in the sense of V. RANDOW [23] (cf. also [16], [17]). V.RANDOW proves in [23], Ch. V, that every orientable Seifert fiber space over S^2 is homeomorphic to a star-shaped tree manifold, and that conversely every star-shaped tree manifold with a certain additional condition allows a Seifert fibration. This fibration is in certain cases uniquely determined by the tree structure or even by the homeomorphism type (cf. also [30]), for example, when the fundamental group is finite and non-cyclic or when the fiber space is a Poincaré space. Therefore, one could have also derived Theorem 2.11 from the results of [28], § 8 and [23], Ch. V. However, given the complexity of the topological arguments, I considered a direct analytical proof desirable.

(d) Following a lemma in [16], a 2-dimenional singularity, whose neighborhood boundary M is a Poincaré space, has a regular resolution. M is thus a tree manifold. If M is star-formed, then so is it in a unique way a fibered Poincaré space.

The fibered Poincaré spaces have been determined in [27], §12. One interesting class of fibered Poincaré spaces, which includes those constructed by DEHN and considered in [27], §13, are the neighborhood boundaries of singularities Σ_{a_1,a_2,a_3}, defined in [5], where the a_i are pairwise coprime.

$$\Sigma_{a_1,a_2,a_3} = \left\{ (z_1, z_2, z_3) \in \mathbb{C}^3 \, \middle| z_1^{a_1} + z_2^{a_2} + z_3^{a_3} = 0, |z| = 1 \right\}.$$

These are Poincaré spaces following [5], p. 6, and they are obviously fibered with 3 exceptional fibers through the operation of $S^1 = \{t \in \mathbb{C} \mid |t| = 1\}$ with

$$(z_1, z_2, z_3) \to (t^{a_2 a_3} z_1, t^{a_1 a_3} z_2, t^{a_1 a_2} z_3).$$

§3. The Icosahedral Singularity

3.1 This paragraph is about the question of which analytic local rings are factorial. It appears that the answer to this question for different dimensions turn out very differently.

For $\dim \mathcal{O}_{X,x} \geqq 4$, one has the following Theorem of GROTHENDIECK (cf. [14], Exposé XI, Cor. 3.14 and [29], §5).

Theorem 3.1 (GROTHENDIECK). *Let (X, x) be a normal complete intersection and $\mathcal{O}_{X,y}$ be factorial for all y except for a 4-codimensional analytic [sub]set. Then, $\mathcal{O}_{X,x}$ is factorial.*

For the case of a isolated hypersurface singularity, one can also deduce this theorem easily from a result of MILNOR (cf. [5], p. 8).

Among the local rings of 3-dimensional hypersurface singularity, there are both infinitely many non-factorial and infinitely many factorial rings. In terms of examples, goes (cf. [5], Korollar 1)

Proposition 3.2. *Let a_1, a_2, a_3, a_4 be natural numbers, one of which is relatively prime to the rest. Then, the following analytic local ring is factorial*

$$\mathbb{C}\{x_1, x_2, x_3, x_4\} \,/\, \left(x_1^{a_1} + x_2^{a_2} + x_3^{a_3} + x_4^{a_4}\right).$$

3.2 Let $\mathcal{O} = \mathcal{O}_{\mathbb{C}^2,0}$ be the local ring of $\mathbb{C}^2$ at 0 and $\widehat{\mathcal{O}}$ its completion. The binary icosahedral group I operates on $\mathbb{C}^2$ with fixed point 0, and hance also on $\mathcal{O}$ and $\widehat{\mathcal{O}}$. Let $\mathcal{O}^{\mathsf{I}}$ resp. $\widehat{\mathcal{O}}^{\mathsf{I}}$ be the subrings of I-invariant elements of $\mathcal{O}$ resp $\widehat{\mathcal{O}}$. Naturally,

$$\mathcal{O}^{\mathsf{I}} \cong \mathcal{O}_{\mathbb{C}^2/\mathsf{I},0} \quad \text{and} \quad \widehat{\mathcal{O}}^{\mathsf{I}} \cong \widehat{\mathcal{O}^{\mathsf{I}}}.$$

KLEIN has determined in [19], Part I, Ch. II, §13 explicitly with invariant theory, three generators for $\mathcal{O}^{\mathsf{I}}$ and the consisting relations between them. It follows

$$\mathcal{O}^{\mathsf{I}} \cong \mathbb{C}\{x, y, z\} / \left(x^2 + y^3 + z^5\right)$$
$$\widehat{\mathcal{O}}^{\mathsf{I}} \cong \mathbb{C}[\![x, y, z]\!] / \left(x^2 + y^3 + z^5\right).$$

Theorem 3.3. *The rings $\mathcal{O}$ and $\mathcal{O}^{\mathsf{I}}$ are, up to isomorphism, the unique factorial two-dimensional analytic local ring.*

Beweis. By Theorems 1.4 and 2.11, $\mathcal{O}$ and $\mathcal{O}^{\mathsf{I}}$ are factorial. Now, conversely, let (X, x) be a 2-dimensional singularity and $\mathcal{O}_{X,x}$ be factorial and not regular. $\mathcal{O}_{X,x}$ is normal as a factorial Noetherian ring. Let $f : X' \to X$ be the minimal resolution of the singularities of X and

$$f^{-1}(x) = \bigcup_{i=1}^{n} X_i$$

where the X_i are the irreducible components. Let $S = (X_i \cdot X_j)$ be the intersection matrix. By Corollary 1.6, (X, x) is rational and $\det S = \pm 1$. Let $Z = \sum r_i X_i$ be the fundamental cycle. Furthermore, cycles $Z_j = \sum r_{ij} X_i$ are defined by the condition $Z_i \cdot X_k = -\delta_{ik}$. From $\det S = \pm 1$ it follows

$$r_{ij} \in \mathbb{Z}. \tag{1}$$

The symmetry of S implies

$$r_{ij} = r_{ji}. \tag{2}$$

Since S is negative definite and $X_i \cdot X_j \geqq 0$ for $i \neq j$, and since $Z_i \cdot X_k \leqq 0$, it follows (cf. [2], p. 130)

$$r_{ij} > 0. \tag{3}$$

The minimality of Z implies due to (1) and (3), that $Z \leqq Z_i$ for $i = 1, \ldots, n$. On the other hand, the relation $Z = -\sum (Z \cdot X_i) Z_i$ with $Z \cdot X_i \leqq 0$ follows due to the definition of the Z_i. Hence, it has to hold for a $k \in \{1, \ldots, n\}$ that $Z = Z_k$, so

$$r_i = r_{lk} \quad \text{for } i = 1, \ldots, n. \tag{4}$$

$Z \leqq Z_j$ implies $r_i \leqq r_{ij}$. Hence, it follows in particular from (2) and (4): $r_k = r_{kk} \leqq r_{kj} = r_{jk} = r_j$ and so

$$r_k \leqq r_j \quad \text{for } j = 1, \ldots, n. \tag{5}$$

By Theorem 1.1 goes

$$Z \cdot Z + Z \cdot K + 2 = 0. \tag{6}$$

By Lemma 1.3 i), the X_i are singularity-free and rational, and therefore

$$X_i \cdot K = -2 - X_i \cdot X_i. \tag{7}$$

The equations (4), (6), (7) and the definition of Z_k imply

$$r_k = 2 - \sum_{i=1}^{n} (X_i \cdot X_i + 2) \, r_i. \tag{8}$$

However, due to the minimality of the resolution f,

$$-(X_i \cdot X_i + 2) \geqq 0. \tag{9}$$

The relations (5), (8), and (9) imply

$$X_i \cdot X_i = -2 \quad \text{for } i = 1, \dots, n. \tag{10}$$

By the classical classification of 2-valued Dynkin diagrams, it follows for the valued graph Γ of the resolution f from (10) and $\det S = \pm 1$

$$\Gamma = \langle 2; 2, 1; 3, 2; 5, 4 \rangle.$$

Hence, it follows from Theorems 2.10 and 2.11

$$(X, x) \cong \left(\mathbb{C}^2/\mathsf{I}, 0\right),$$

and so

$$\mathcal{O}_{X,x} \cong \mathcal{O}^{\mathsf{I}}.$$

q.e.d. □

3.3 (a) While so far in this work only analytic local rings, i.e. local rings of complex spaces, have been considered, in this last section some statements about other factorial local rings shall be derived from the main theorem 3.3. There, I have used not-easy-to-prove comparable theorems of ARTIN and HIRONAKA. The following proofs likely allow simplification.

Corollary 3.4. *The local rings $\widehat{\mathcal{O}}$ and $\widehat{\mathcal{O}}^{\mathsf{I}}$ are, up to isomorphism, the unique factorial complete two-dimensional local rings with residue field $\mathbb{C}$.*

Beweis. Let R be a factorial complete 2-dimensional local ring with residue field $\mathbb{C}$. Then R is normal as a factorial Noetherian ring. According to ARTIN, every complete normal 2-dimensional local ring with a residue field k of characteristic 0 is the completion of a local ring of an algebraic surface over k (see [2′], p.36). Thus, let R_a be an algebraic-geometric local ring over $\mathbb{C}$ with $R = \widehat{R}_a$. According to GAGA [30], Prop.3, one can, in a canonical way, embed R_a into an analytic local ring of the same dimension R_h, so that it holds for the completion $\widehat{R}_a = \widehat{R}_h$. R_h is, according to MORI, factorial, since $\widehat{R}_h$ is factorial (cf. e.g. [3], §3, n^0 7, Prop. 4). Thus, by Theorem 3.3 follows $R_h = \mathcal{O}^{\mathsf{I}}$ and therefore $R = \widehat{R}_h = \widehat{\mathcal{O}}^{\mathsf{I}}$. That $\widehat{\mathcal{O}}^{\mathsf{I}}$ is factorial follows e.g. from [25], Satz 1. □

(b) SAMUEL has stated the following conjecture in [24], p. 171: If R is a complete factorial local ring, the formal power series ring $R[\![X]\!]$ is factorial. Examples by SAMUEL and others show that the assumption of completeness cannot be omitted. To my knowledge, the conjecture has not yet been fully decided. For example, it is correct for R with codh $R \geqq 3$ (SCHEJA [26], Theorem 2). For dim $R = 2$, Corollary 3.4 shows that the conjecture is correct under the additional assumption that the residue class field is $\mathbb{C}$. SCHEJA proves in [26], Theorem 4, that $\widehat{\mathcal{O}}^{\mathsf{I}}[\![X]\!]$ is factorial.

(c) There are many algebraic-geometric local rings over $\mathbb{C}$, i.e. here: localizations of a finitely generated algebra over $\mathbb{C}$ with respect to a maximal ideal, which have dimension 2 and are factorial. In terms of examples, according to SAMUEL, the 2-dimensional algebraic-geometric local rings

$$\left(\mathbb{C}\left[x_1, x_2, x_3\right] / \left(x_1^{a_1} + x_2^{a_2} + x_3^{a_3}\right)\right)_{\mathfrak{m}},$$

where $\mathfrak{m}$ is the maximal ideal generated by x_1, x_2, x_3, are factorial, if the natural numbers a_i are pairwise coprime (cf. [3], §3, p. 99, Exercise 7 and §3, n^0 4, prop. 3). However, the completion of these local rings

$$\mathbb{C}[\![x_1, x_2, x_3]\!] / \left(x_1^{a_1} + x_2^{a_2} + x_3^{a_3}\right)$$

are by Corollary 3.4 not factorial, if $a_i > 1$ and $\{a_1, a_2, a_3\} \neq \{2, 3, 5\}$. The following corollary shows, that the factoriality first gets lost not during completion, but rather already during Henselization.

Corollary 3.5. *Let the 2-dimensional local ring R be the Henselization of an algebraic-geometric local ring over $\mathbb{C}$. Then, R is factorial exactly if the completion $\widehat{R}$ is isomorphic to $\widehat{\mathcal{O}}$ or $\widehat{\mathcal{O}}^{\mathsf{I}}$.*

Beweis. Let R be the Henselization $\widetilde{R}_a$ of the algebraic-geometric local ring R_a.R_a is Noetherian and therefore $\widetilde{R}_a$ is also Noetherian, and since it is also factorial, $\widetilde{R}_a$ is normal. Then $\widehat{R}_a$ and the analytic local ring R_h corresponding to R_a are also normal. Therefore, the spectra of these local rings outside the closed points $\widetilde{\mathfrak{m}}$ resp. $\mathfrak{m}_h$ resp. $\hat{m}$ are regular, and therefore (cf. [14], ExposéXI, Cor.3.8) the divisor class groups of these local rings are the same as the Picard groups of the ***gelochten spectra. According to an unpublished result by HIRONAKA, the following generally applies for a local ring R_a of an isolated singularity of an algebraic variety over $\mathbb{C}$

$$\operatorname{Pic}\left(\operatorname{Spec}\left(\widetilde{R}_a\right) - \widetilde{\mathfrak{m}}\right) = \operatorname{Pic}\left(\operatorname{Spec}\left(R_h\right) - \mathfrak{m}_h\right) = \operatorname{Pic}\left(\operatorname{Spec}\left(\widehat{R}_a\right) - \widehat{\mathfrak{m}}\right).$$

R is thus factorial, exactly if $\widehat{R}$ is factorial, and therefore the corollary follows trivially from 3.4. □

Addition to the correction. Under the use of a theorem of ARTIN, 3.5 can be improved in the following manner. If the Henselization of a 2-dimensional algebraic-geometric local ring over $\mathbb{C}$ is factorial, then it is isomorphic to the Henselization of $\mathbb{C}[x, y]_{(x,y)}$ or of $\left(\mathbb{C}[x, y, z] / x^2 + y^3 + z^5\right)_{(x,y,z)}$.

JOSEPH LIPMAN has shared with me, that at the same time of the following generalizations of results of this work, he has obtained: Corollary 1.6 holds for each excellent henselian 2-dimensional normal local ring R with algebraically closed residue field k. If such an R is thus factorial, then the minimal resolution has by §3 the valued graph $\langle 2;2,1;3,2;5,4\rangle$. For char $k \neq 2,3,5$, the maximal ideal $\mathfrak{m}$ is generated by a system of parameters x, y, z with $x^2 + y^3 + z^5 = 0$, and an analogue to Corollary 3.4 holds therefore, under these general requirements.

Bibliography

1. ARTIN, M.: Some numerical criteria for contractability of curves on algebraic surfaces. Amer. J. Math. **84**, 485 – 496 (1962).

2. —- On isolated rational singularities of surfaces. Amer. J. Math. **88**, 129 – 136 (1966).
   2′. —- On algebraic extensions of local rings. Rend. Mat. e. Appl. **25**, 33 – 37 (1966).

3. BOURBAKI, N.: Algèbre commutative, chapitre 7, Diviseurs. Paris: Herman 1965.

4. BRIESKORN, E.: Über die Auflösung gewisser Singularitäten von holomorphen Abbildungen. Math. Ann. 166, 76 – 102 (1966).

5. —- Beispiele zur Differentialtopologie von Singularitäten. Inventiones math. **2**, 1 – 14 (1966).

6. BRODY, E.J.: The topological classification of the Lens spaces. Ann. of Math. **71**, 163 – 184 (1960).

7. CARTAN, H.: Quotient d'un espace analytique par un groupe d'automorphismes. In Älgebraic geometry and topology", p.90 – 102. Princeton: Princeton University Press 1957.

8. DUVAL, P.: Homographies, quaternions and rotations. Oxford: Oxford University Press 1964.

9. FOX, R.H.: Covering spaces with singularities. In Älgebraic geometry and topology', p. 243 – 257. Princeton: Princeton University Press 1957.

10. GODEAUX, L.: Note sur les involutions cycliques appartenants à une surface algébrique. Arch. Math. **6**, 1 – 4 (1955).

11. GOTTSCHLING, E.: Die Uniformisierbarkeit der Fixpunkte eigentlich diskontinuierlicher Gruppen von biholomorphen Abbildungen. Math. Ann. **169**, 26 – 54 (1967).

12. GRAUERT, H., R. REMMERT: Komplexe Räume. Math. Ann. **136**, 245 – 318 (1958).

13. GROTHENDIECK, A.: Étude locale des morphismes; germes d'espaces analytiques, platitude, morphismes simples. Exposé 13 in Séminaire Cartan 13 (1960/61).

14. —- Séminaire de Géométrie Algébrique 1962.

15. HIRZEBRUCH, F.: Über vierdimensionale Riemannsche Flächen mehrdeutiger analytischer Funktionen von zwei komplexen Veränderlichen. Math. Ann. **126**, 1 – 22 (1953).

16. —- The topology of normal singularities of an algebraic surface. Séminaire Bourbaki, 15^e année 1962/63, No. 250.

17. —- Über Singularitäten komplexer Flächen. Rend. Mat. e. Appl. **25**, 213 – 232 (1966).

18. HOLMANN, H.: Seifertsche Faserräume. Math. Ann. **157**, 138 – 166 (1964).

19. KLEIN, F.: Vorlesungen über das Ikosaeder und die Auflösung der Gleichung vom fünften Grade. Leipzig 1884.

20. KOESTNER, A.: Lokale Darstellbarkeit eindimensionaler analytischer Mengen in zweidimensionalen normalen komplexen Räumen durch eine holomorphe Funktion. Schriftenreihe Math. Inst. Univ. Münster, Münster 1967.

21. MUMFORD, D.: The topology of normal singularities of an algebraic surface and a criterion for simplicity. Publ. Math. de l'Institut des Hautes Études Scientifiques No. 9, Paris 1961.

22. PRILL, D.: Local classification of quotients of complex manifolds by discontinuous groups. Duke Math. J. **34**, 375 – 386 (1967).

23. RANDOW, R. V.: Zur Topologie von dreidimensionalen Baummannigfaltigkeiten. Bonner Math. Schriften Nr. 14, Bonn 1962.

24. SAMUEL, P.: Sur les anneaux factoriels. Bull. Soc. Math. France **89**, 155 – 173 (1961).

25. SCHEJA, G.: Uber Primfaktorzerlegung in zweidimensionalen lokalen Ringen. Math. Ann. **159**, 252 – 258 (1965).

26. —- Einige Beispiele faktorieller lokaler Ringe. Math. Ann. **172**, 124 – 134 (1967).

27. SEIFERT, H.: Topologie dreidimensionaler gefaserter Räume. Acta Math. **60**, 147 – 238 (1933).

28. —-, U. W. THRELFALL: Topologische Untersuchung der Diskontinuitätsbereiche endlicher Bewegungsgruppen des dreidimensionalen sphärischen Raumes (Schluß). Math. Ann. **107**, 543 – 586 (1932).

29. STORCH, U.: Fastfaktorielle Ringe. Schriftenreihe Math. Inst. Univ. Münster, Münster 1967.

30. SERRE, J.P.: Géométrie algébrique et géométrie analytique. Ann. Inst. Fourier **6**, 1 – 42 (1956).

31. WALDHAUSEN, F.: Eine Klasse von 3-dimensionalen Mannigfaltigkeiten I. Inventiones math. **3**, 308 – 333 (1967).

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