

3264 and Algebraic Cobordism

Michael Ruofan Zeng 曾若凡

Based on joint work with Arkamouli Debnath, arXiv:2604.14536:

"Oriented cohomology rings of some moduli spaces via blowups."

University of Washington, Seattle

0.

Steiner's 5 conics problem

Problem:

How many conics are simultaneously
tangent to 5 general conics in $\mathbb{P}^2$?

Solving an enumerative problem

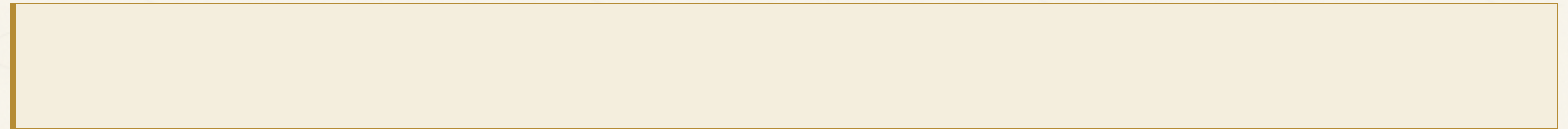
5 steps of solving an enumerative problem

1. Choose a smooth projective parameter space.
2. Determine its Chow ring.
3. Express each condition as a cycle class.
4. Multiply the classes.
5. Check transversality.

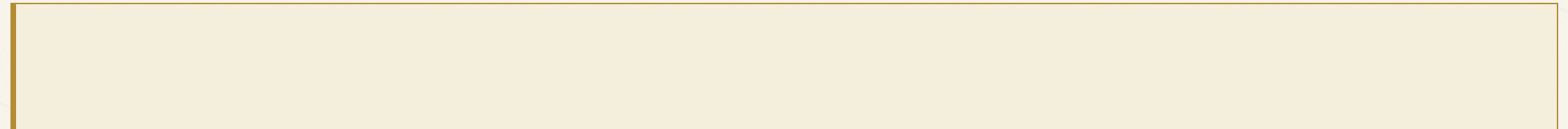
Step 1 · Parameter space

Fix five general smooth conics $C_1, \dots, C_5$ in $\mathbb{P}^2$, with k algebraically closed of characteristic zero.

A conic is a nonzero quadratic equation $C(x, y, z) = 0$, up to scaling.



The parameter space is



Step 2 · Chow ring

Let $\mathcal{H}$ be the hyperplane class. The Chow ring is

$$A^*(\mathbb{P}^4) = \mathbb{Z}[\mathcal{H}] / (\mathcal{H}^5)$$

Five general hyperplanes meet in a point:

Counting solutions

Tangency of plane conics

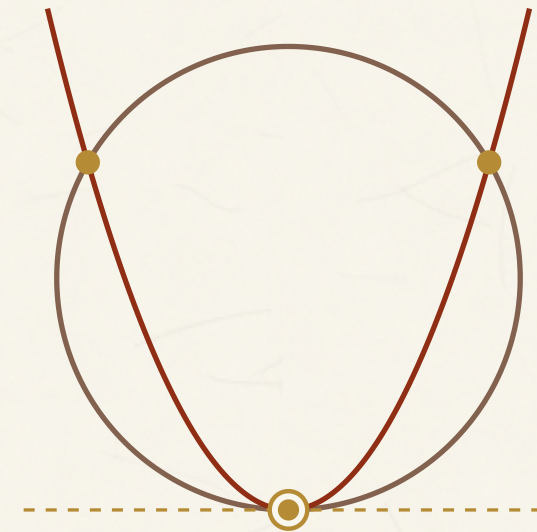
Definition · Tangent conics

Let C_1 and C_2 be smooth conics with no common component. They are **tangent at** P if

They are **tangent** if this holds at some intersection point.

Equivalently, their tangent lines at P agree.

An ordinary tangency has one double intersection and two further simple intersections:



A common tangent at

Step 3 · Chow class of the tangency locus

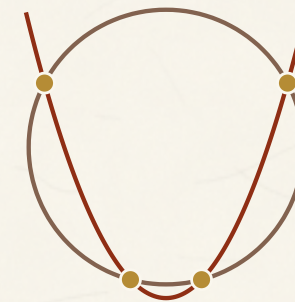
Let $\mathcal{C}$ be the tangency locus. A general line in $\mathcal{C}$ is a pencil of quadratic equations.

On $\mathcal{C}$, the pencil has no base points and defines a degree-four map:

Its fibers are the four intersection points in the picture. Tangency gives a branch value. Riemann–Hurwitz gives

where $\mathcal{R}$ is the ramification divisor. For a general pencil these are six simple tangencies at distinct parameters.

Four distinct intersection points



Two points meet: tangency



— A conic in the pencil

Step 4 · Intersection product in the Chow ring

All five tangency conditions have class σ . Multiplying in the Chow ring gives

$$\sigma^5 = 5\sigma^3\tau = 5\sigma^2\tau^2 = 5\sigma\tau^3 = 5\tau^4$$

conics, given that σ and τ are generically transverse!

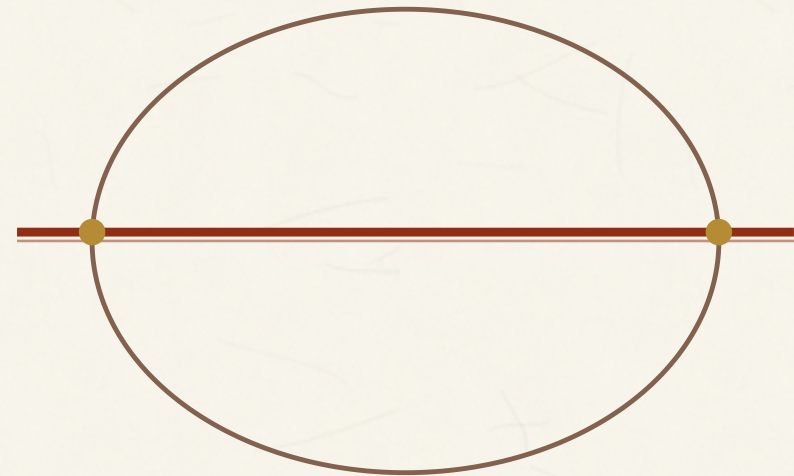
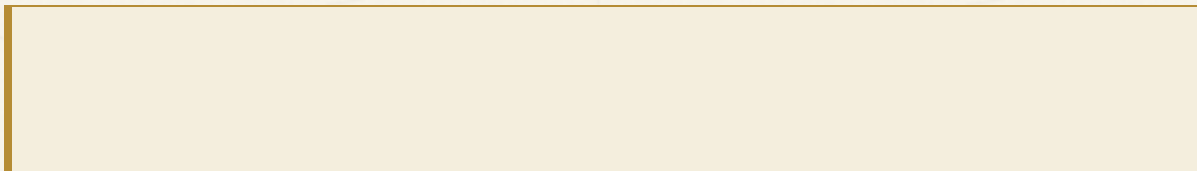
Step 5 · Transversality

Let ℓ be a linear form. The restriction of a double line to every smooth conic is a square:

If the line meets C transversely at P, Q , the double line has intersection divisor $2P + 2Q$. Thus it satisfies the tangency condition at both points.

All five tangency loci contain the Veronese surface

:

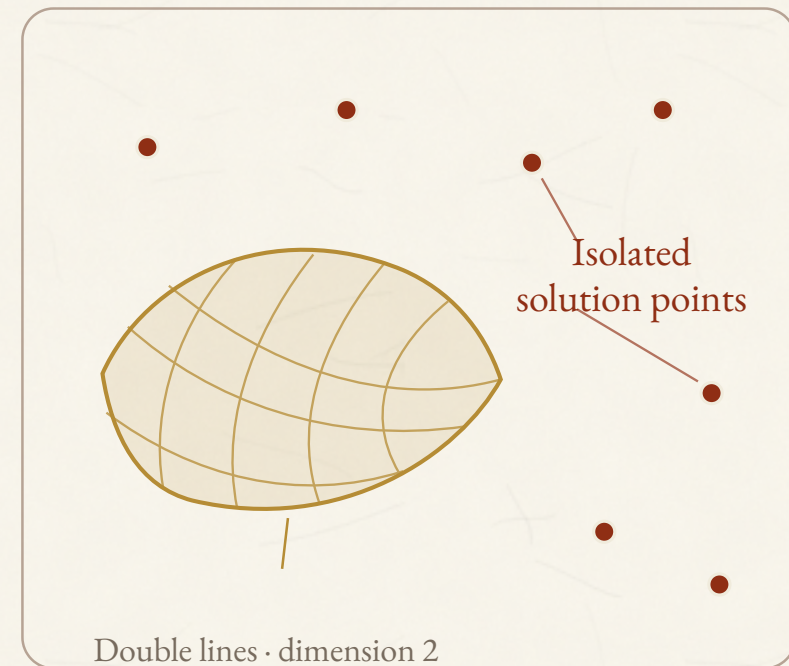


Step 5 · Trouble!

Squaring linear forms gives the Veronese embedding of the plane parametrizing lines:

For five general given conics, the common zero locus contains this surface and the isolated solutions:

Excess Intersection!!



Schematic support; only a few points are drawn.

Step I+ · The moduli space of complete conics

Definition · Moduli space of complete conics

The **moduli space of complete conics** is the blowup

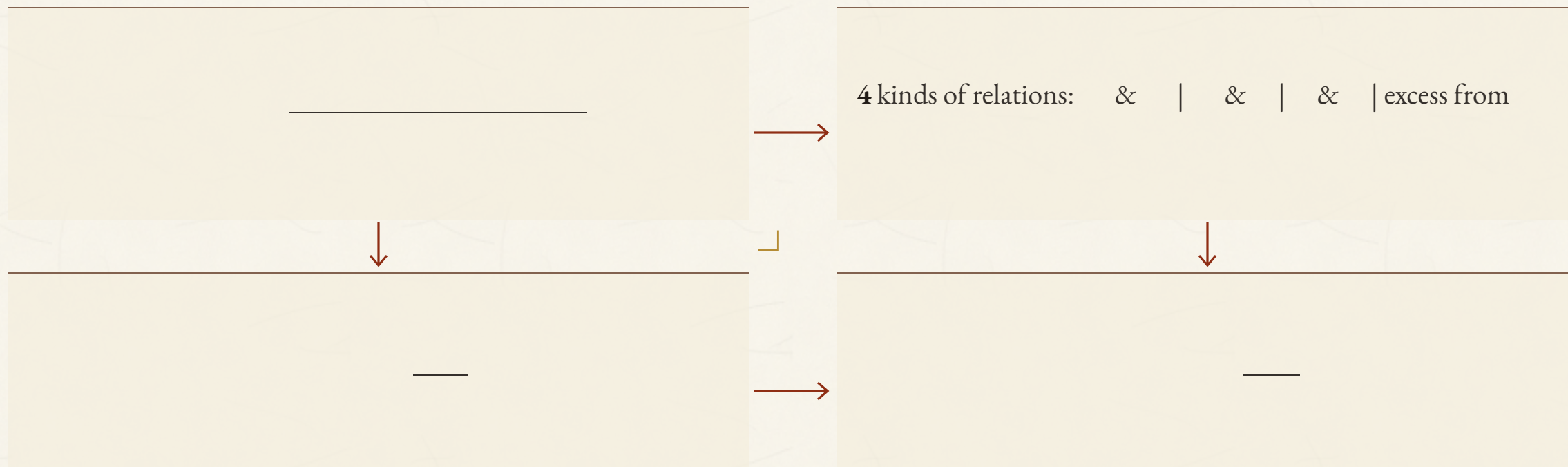
where V is the Veronese surface parametrizing double lines.

Its exceptional divisor is the **projective normal bundle**



Step 2+ · The Chow ring of the moduli space of complete conics

Here σ , τ , and ρ are the hyperplane classes on $\mathbb{P}^2$, $\mathbb{P}^1$, and $\mathbb{P}^1$, respectively. We also write σ^* , τ^* , and ρ^* for their pullbacks by π .



Step 2+ · The Chow ring of the moduli space of complete conics

On $\mathcal{M}_2$, let π be the pullback of the hyperplane class from $\mathbb{P}^2$ on $\mathcal{M}_2$ and σ be the class of a point. Then, $\pi^2 = 2\sigma$ for $\mathcal{M}_2$, so that $\pi^3 = 0$. Let κ be the

$$\left(\begin{array}{c} \pi^2 \\ \pi \sigma \\ \sigma^2 \end{array} \right)$$

Here, σ is the class of a point.

Step 3+ · Class of the tangency locus

Perturb a double line by a general quadratic form Q , with parameter t . The discriminant vanishes to order two; Q is its leading coefficient.

The strict transform $\tilde{C}$ therefore removes twice the exceptional divisor:

$$C \sim 2E + \tilde{C}$$

Taking divisor classes in $\tilde{C}$ gives

$$\sim$$

Step 4 · Intersection product in

Theorem 5.12 (Chasles 1864)

There are ∞^2 conics simultaneously tangent to
general conics in $\mathbb{P}^2$, which are not double lines.

Macaulay2 · integral Chow ring

```
R = ZZ[symbol e00,symbol e10,
      symbol e20,symbol alpha,
      MonomialOrder=>Lex];
CH = R/ideal(alpha^6,
  alpha*e00-2*e10,alpha*e10-2*e20,
  alpha*e20,e10^2-e00*e20,
  e10*e20,e20^2,
  e00^3-9*e00*e10+30*e20-4*alpha^3,
  e00^2*e10-9*e00*e20-2*alpha^4,
  e00^2*e20-alpha^5);
use CH;
solution = (6*alpha-2*e00)^5;
coefficient(R_3^5, lift(solution,R))
```

[View output](#)
[Full script](#)

Recorded Macaulay2 output

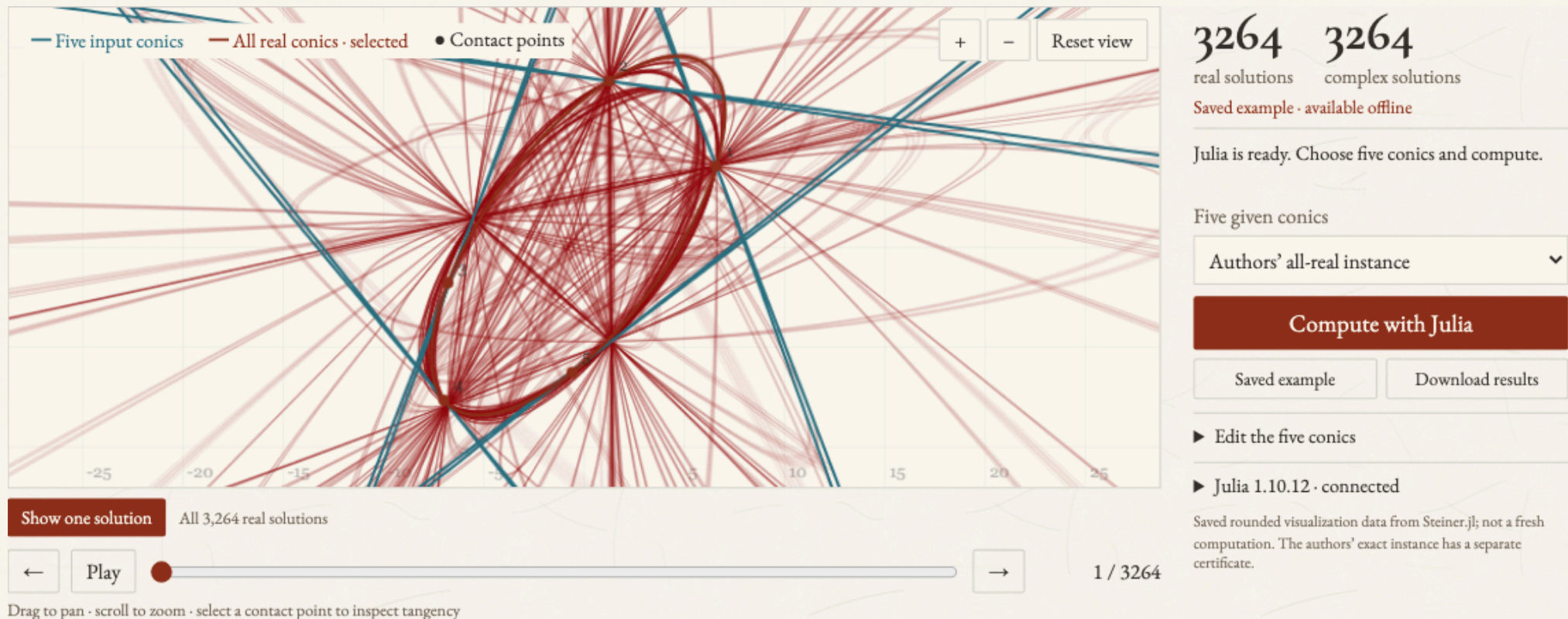
Problem:

Can all 3,264 conics tangent to five general real conics be real?

LIVE JULIA DEMO

3264 Conics in a Second

Paul Breiding, Bernd Sturmfels, Sascha Timme. **3264 conics in a second**. Notices Am. Math. Soc. 67, No. 1, 30-37 (2020).



Explore the saved solutions in the online slides.

Projective bundle
formula

Fundamental class

Functoriality

Projection formula

Transverse base change

Transverse intersection

First Chern class of

Whitney sum formula

Self-intersection

Excess intersection

Dimension

Blowup Formula

1.

Oriented cohomology theories

Oriented cohomology theories

Let $\mathcal{C}$ be the category of smooth k -schemes.

Definition 2.1 · 1/2

An *oriented cohomology theory* on $\mathcal{C}$ is a pair (h, σ) consisting of:
an additive contravariant functor $h: \mathcal{C} \rightarrow \mathcal{A}$

Oriented cohomology theories

Definition 2.1 · 2/2

for each *projective* morphism $f: X \rightarrow Y$ in $\mathcal{C}$ of relative dimension d , a graded R -module homomorphism

where the R -module structure on $H_*^f(X, Y)$ is induced by f^* . They satisfy the axioms discussed in the following slides.

Projective bundle formula

Convention: $\mathbb{P}^n$ parametrizes lines, with tautological bundle $\mathcal{O}(-1)$.

Theorem 2.3 (Levine–Morel 2007)

Let E be a vector bundle of rank r over X . The oriented cohomology ring of the projective bundle $\mathbb{P}(E)$ has the quotient presentation

where ξ is the $\mathbb{A}^1$ -theoretic hyperplane class of $\mathbb{P}(E)$.

Fundamental classes

Definition 2.17

Let X , and let $i: Y \rightarrow X$ be a regular closed immersion of codimension d with Y smooth. The *fundamental class* of Y in X is the $\mathbb{A}^1$ -cohomology class

where γ_Y is the Gysin morphism attached to i .

Since $\mathbb{A}^1(X)$ is a unital ring, it has a unit which we denote by 1_X .

Divisor classes and transverse intersection

Proposition 2.20 (Levine–Morel 2007)

Let X and let D be a smooth divisor. Then,

Proposition 2.18 (Déglise–Jin–Khan 2021)

Let X . Suppose V and W are closed immersions of smooth subschemes of X of codimensions c_V and c_W . Further assume V and W are transverse and X is smooth of codimension c in Y . Then, the identity

holds in $A_{*}(X)$.

Formal group law

For line bundles $L_1, \dots, L_n$ on X , a formal group law λ computes the first Chern class of their tensor product:

Here $x_1, \dots, x_n$ are formal variables, and the coefficients lie in the theory's coefficient ring:

Chow and periodic K -theory have different group laws:

Algebraic cobordism [LM07] carries the **universal formal group law**

Formal group law

For the formal group law F , write F^{-1} . The F -series $F^{-1}(x)$ is the F -fold sum under F . The formal inverse F^{-1} satisfies $F(x, F^{-1}(x)) = 0$; set $F^{-1}(0) = 0$.

For a Cartier divisor D , its fundamental class is $[D]$.

Proposition 2.22 (Levine–Morel 2007)

Let X and let D_1 and D_2 be effective Cartier divisors on X . Then,

where F is the formal group law of X .

Dualizing a line bundle gives the formal inverse:

2.

Motivic spectra

The stable motivic homotopy category

Definition · Stable motivic homotopy category

A **pointed motivic space** is a presheaf of pointed spaces on $\mathcal{M}_*^{\text{fp}}$ satisfying Nisnevich descent and $\mathbb{A}^1$ -homotopy invariance for the projection $\mathbb{A}^1 \times B \rightarrow B$:

$\longrightarrow$

Write $\mathcal{M}_*^{\text{fp}}$ for their category. The **stable motivic homotopy category** is its stabilization with respect to $\mathbb{A}^1$:

Oriented motivic ring spectra

Let $\mathcal{A}$ denote the algebraic cobordism spectrum.

Theorem-Definition (Panin–Pimenov–Röndigs 2008; Déglise–Jin–Khan 2021)

A $\mathcal{A}$ -ring spectrum E is *oriented* if it receives a morphism of motivic ring spectra $\mathcal{A} \rightarrow E$.

Let E be a motivic ring spectrum in $\mathcal{M}^{\text{cts}}$. For $n \in \mathbb{Z}$ and $k \in \mathbb{N}$, set

Then, the functor $E_{\mathcal{A}} : \mathcal{M}^{\text{cts}} \rightarrow \mathcal{A}b$ is an oriented cohomology theory satisfying the aforementioned axioms.

Oriented motivic ring spectra

The **Chow line** means

. For smooth in characteristic zero:

Theory

Spectrum

Chow theory

Periodic -theory

Algebraic cobordism

Oriented motivic ring spectra

Fix a prime p ; τ_n denotes the n th Tate twist. For Betti cohomology, fix an embedding $\iota: \mathbb{Q} \hookrightarrow \mathbb{C}$.

Theory

Degree- n group

Even étale cohomology

Even algebraic de Rham cohomology

Even Betti cohomology

All the above are

$\mathbb{A}^1$ -invariant cohomology theories.

Problem: Hodge cohomology

For smooth X , **Hodge cohomology** consists of the groups

where Ω^i_X is the sheaf of differential i -forms. For $X = \mathbb{A}^n$, this is the ring of global regular functions.

Pullback along f gives

$$\underbrace{f^* \Omega^i_X}_{\cong \Omega^i_Y} \cong \Omega^i_Y$$

Hodge cohomology fails $\mathbb{A}^1$ -invariance and is not represented in $\mathcal{M}^c$!

Motivic spectra: non- $\mathbb{A}^1$ -invariant generalization

Let S be a base scheme. Write $\mathcal{S}m_S$ for smooth S -schemes and $\mathcal{S}p_S$ for spectra.

Definition · Elementary blowup excision (ebu) (Annala–Iwasa 2022)

A Zariski sheaf $\mathcal{F}$ satisfies **elementary blowup excision** if applying $\mathcal{F}$ to the displayed square gives a homotopy Cartesian square for every integer n and every X .

As a sheaf, $\mathcal{F}_n(X) = \mathcal{F}_n(X, \mathbb{A}^1_X)$, where $\mathbb{A}^1_X$ is the zero spectrum.

Here $\mathbb{A}^1_X$ is the zero section, and $\mathbb{A}^1_X$ is the exceptional divisor.



Motivic spectra

Definition · Motivic spectra (Annala–Hoyois–Iwasa 2025)

The category of **motivic spectra** is

The broader category includes theories without $\mathbb{A}^1$ -invariance:

Theory

Notation

Hodge cohomology

Topological Hochschild homology

Topological cyclic homology

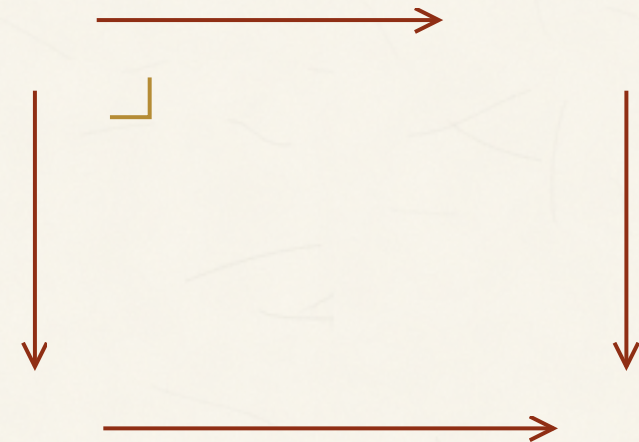
Motivic spectra

Let $i: X \hookrightarrow Y$ be a closed immersion in $\mathcal{H}_k$, with exceptional divisor E .

Definition (Annala–Hoyois–Iwasa 2025)

A presheaf $\mathcal{H}$ satisfies **smooth blowup excision** if $\mathcal{H}(X) \cong \mathcal{H}(Y)$ and applying $\mathcal{H}$ to any blowup square gives a homotopy Cartesian square for **every** such closed immersion.

With Nisnevich descent, (Annala–Hoyois–Iwasa 2025); with an orientation, (Annala–Iwasa 2022). Together:



Motivic spectra

The same pointed projective line is inverted in both constructions (Annala–Hoyois–Iwasa 2025, §4).

Descent	Nisnevich	Zariski; Nisnevich is additional
-invariance	Required	Not imposed
Blowup excision	Consequence	ebu; sbu with Nisnevich descent
Stabilization	invertible	invertible
Examples		Also Hodge cohomology,

Imposing Nisnevich descent and -invariance recovers:

3.

The blowup formula

Additive Blowup Formula

Again, let X be smooth and
of codimension c .

i a closed embedding



Additive blowup formula

Theorem 3.3 (Debnath–Z.)

Let $\mathcal{R}$ be an oriented motivic ring spectrum in $\mathcal{M}_\mathbb{A}^{\text{sm}}$ satisfying smooth blowup excision and the projective bundle formula. Let $i: Y \hookrightarrow X$ be a regular closed immersion of codimension d and let $\text{Bl}_i X$ be the blowup with exceptional divisor E . Then, for every $n \in \mathbb{Z}$, there is a direct sum decomposition

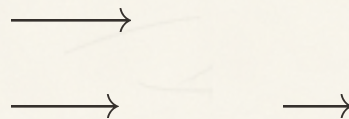
Examples of these higher $\mathcal{R}$ -groups include higher Chow groups and higher K -groups.

Additive blowup formula

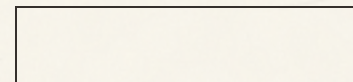
Under Theorem 3.3, let $\square$.
Smooth blowup excision applies to this square.



There is a long exact sequence



It suffices to construct a **splitting** in every bidegree:



Then $\square$, and the resulting short exact sequences split.

Additive blowup formula

In $\mathcal{A}^*$, **homotopy purity** gives the collapse map for $\mathcal{A}^*$:

$$\mathcal{A}^* \longrightarrow \mathcal{A}^*$$

Applying cohomology, the Thom isomorphism defines $\mathcal{A}^*$ as the composite

$$\mathcal{A}^* \longrightarrow \mathcal{A}^* \longrightarrow \mathcal{A}^*$$

Put $\mathcal{A}^*$. **Self-intersection** gives the lifts

The **projective bundle formula** then gives the splitting.

In $\mathcal{A}^*$, **purity fails in general**. For $\mathcal{A}^*$,

would be an equivalence only if $\mathcal{A}^*$ were one [Ta26].

Additive blowup formula

Longke Tang constructed a **Thom collapse map** in .



The **Gysin pushforward** is . Since is the zero-section map,

The **projective bundle formula** implies every can be written uniquely as

Self-intersection formula lifts every term having :

These lifts define the desired **splitting**:

Oriented cohomology ring of

Theorem 3.9 (Blowup Presentation Formula, Debnath–Z.)

Work with an $\mathbb{A}^1$ -invariant oriented theory $\mathbb{H}$. Let $\pi: \tilde{X} \rightarrow X$ be a smooth closed embedding of codimension r , E the blowup, $\mathcal{O}(1)$ the exceptional divisor, and $p: \mathbb{P}^r \rightarrow \text{pt}$ the projection. Let ξ be the relative hyperplane class of the projective bundle $\mathbb{P}^r$ in $\mathbb{H}$ -theory. The ring $\mathbb{H}(\tilde{X})$ is generated over $\mathbb{H}(X)$ by pullbacks π^* for $\alpha \in \mathbb{H}(X)$ and pushforwards π_* for $\beta \in \mathbb{H}(\mathbb{P}^r)$ subject to the following families of relations.

(i) (*Intersection on $\mathbb{P}^r$*) For $\beta \in \mathbb{H}(\mathbb{P}^r)$, $\pi_* \pi^* \beta = \beta$, (ii) (*Intersection between $\mathbb{P}^r$ and E*) For $\beta \in \mathbb{H}(\mathbb{P}^r)$ and $\gamma \in \mathbb{H}(E)$,

(iii) (*Intersection on E*) For $\gamma \in \mathbb{H}(E)$, (iv) (*Excess intersection formula*) For every $\gamma \in \mathbb{H}(E)$, we have

where $\pi^* \pi_*$ is the formal inverse of $\pi_* \pi^*$ under the group law $1 + \pi^* \pi_*$.

where

Oriented cohomology ring of

Theorem 3.10

Assume in addition that the pullback Chern class $\sim$. Choose lifts $\sim$ for $\sim$ is surjective. Let $\sim$ denote the first Chern class in $\sim$.
 Then the canonical map $\sim$ given by $\sim$ and $\sim$ induces an isomorphism $\sim$.

The resulting quotient is independent of the choices of the lifts $\sim$.

Here, $\sim$ is the fundamental class of the exceptional divisor in $\sim$.

4.

Secant lines to the twisted cubic

Problem:

How many secant lines to a twisted cubic
in meet two general lines?

Secants to a twisted cubic

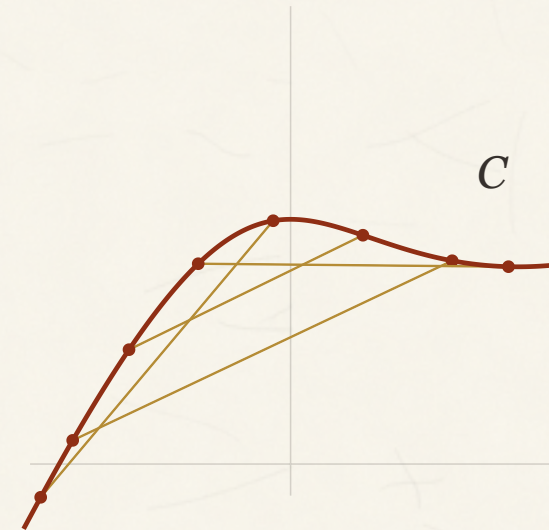
Let C be the twisted cubic.

Parameter space:

The blowup is the

Universal secant line

A point of $\mathcal{U}$ records a secant together with a point on it.



The twisted cubic

Oriented cohomology ring of

Let $\mathcal{H}$ be an oriented cohomology theory with coefficient ring R and formal group law

Let X be a smooth variety, $\mathcal{H}(X)$ its oriented cohomology ring, and $\mathcal{H}(X)$ on X . Then

Steps to applying the full blowup presentation formula:

1. Compute the **normal bundle** $N_{X/Y}$.
2. Use the **projective bundle formula** for $\mathcal{H}(P(N_{X/Y}))$, where $P(N_{X/Y})$ is the projective bundle of lines in $N_{X/Y}$; compute powers and the formal inverse of β .
3. Compute pushforwards from the blowup center X in preparation of excess intersection.
4. Take **pullbacks from** $\mathcal{H}(P(N_{X/Y}))$ and **pushforwards from** $\mathcal{H}(X)$ as generators, and impose relations (i)–(iv).

Oriented cohomology ring of

Normal bundle of

Proposition 4.3 (Ghione–Sacchiero 1980)

Let ι be the cubic Veronese embedding, equivalently the twisted cubic $\mathbb{P}^3 \rightarrow \mathbb{P}^4$. Then,

We can compute its Chern classes. Since $\mathbb{P}^4$ is a complex manifold, the formal-group terms in $H^*(\mathbb{P}^4; \mathbb{Z})$ vanish:

Oriented cohomology ring of

Let $\pi: \mathbb{P}^n \rightarrow \mathbb{P}^1$, write $\pi^* \mathcal{O}(1)$ also for its pullback to $\mathbb{P}^n$, and set $\mathcal{O}(1) = \pi^* \mathcal{O}(1)$.

Projective bundle formula:

$$H^*(\mathbb{P}^n, \mathbb{Z}) \cong H^*(\mathbb{P}^1, \mathbb{Z}) \oplus \bigoplus_{i=1}^n H^*(\mathbb{P}^1, \mathbb{Z}) \otimes \mathcal{O}(i)$$

Powers and formal inverse of $\mathcal{O}(1)$:

Oriented cohomology ring of

Fundamental classes from

Let π and i be the structural maps. Since $i^* \pi^* = \text{id}$, write $\pi^* = i_* i^*$.
 Quillen's pushforward formula gives $i_* i^* = \text{id}$ and $i_* i^* = \text{id}$. Thus,

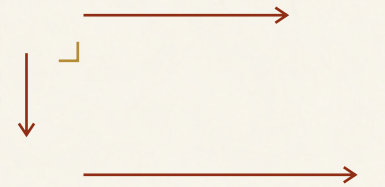
Proposition 4.6

The $\mathbb{A}^1$ -theoretic fundamental class of the twisted cubic in $\mathbb{A}^3$ is $\pi_* i^* \pi^* \pi^*$.

Oriented cohomology ring of

4 kinds of relations:

Oriented cohomology ring of

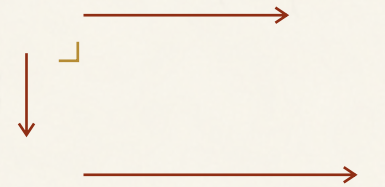


(i) Intersection on

Pullback is a ring hom:

The only pullback relation is therefore

Oriented cohomology ring of



(ii) Intersection between and

For , use projection formula, , and abusing notation:

Applying this to gives

In particular, .

Oriented cohomology ring of



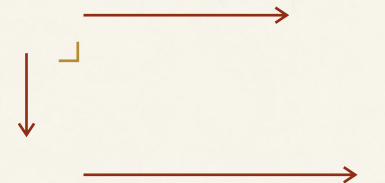
(iii) Intersection on

Self-intersection formula: for any ,

Exhausting the combinations gives

The rest are zero since .

Oriented cohomology ring of



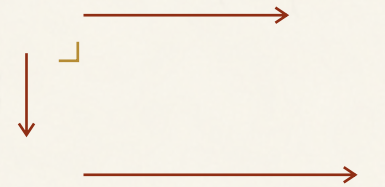
(iv) Excess intersection

The normal bundle has rank two, so the excess bundle is a line bundle. It is defined by

Whitney sum formula gives

Family (iv) is therefore

Oriented cohomology ring of



(iv) Excess intersection

It suffices to evaluate on σ_1 and σ_2 , the two basis classes of $H^2(X)$. First, σ_1 because σ_1 represents a point of X :

For σ_2 , substitute the fundamental class σ_2 :

Thus σ_1 is eliminated, and the remaining excess relation is

Oriented cohomology ring of

Theorem 4.8 (Debnath–Z.)

Let $\mathcal{H}$ be an oriented cohomology theory. Let $\mathcal{H}^*(\mathbb{A}^3)$ be the twisted cubic and $\mathcal{H}^*(\mathbb{A}^3) \hookrightarrow \mathcal{H}^*(\mathbb{A}^3)$ be the inclusion map, let $\pi: \mathcal{H}^*(\mathbb{A}^3) \rightarrow \mathcal{H}^*(\mathbb{A}^3)$ be the blowup, let $\mathcal{H}^*(\mathbb{A}^3)$ be the exceptional divisor, and let $\mathcal{H}^*(\mathbb{A}^3)$ be the projection. With slight abuse of notation, let $\mathcal{H}^*(\mathbb{A}^3)$ and $\mathcal{H}^*(\mathbb{A}^3)$ where $\mathcal{H}^*(\mathbb{A}^3)$. Then, there is an isomorphism of graded $\mathcal{H}^*(\mathbb{A}^3)$ -algebras

We did it!!

Secants to a twisted cubic

The three quadrics through ℓ define the morphism

Let π be the pullback of a general hyperplane in $\mathbb{P}^3$. Then,

For a general line ℓ , these quadrics restrict to the complete system $\mathcal{Q}_\ell$. Thus the locus of secants meeting ℓ is a plane conic.

Secants to a twisted cubic

For a general line ℓ , let D_ℓ be the divisor of marked secants whose line meets ℓ . Since ℓ is a conic,

Here D_ℓ and $D_{\ell'}$. Unlike D_ℓ , we need to use the formal group law of $\mathcal{C}$:

Secants to a twisted cubic

Multiply the two incidence classes and the hyperplane class in
:

Theorem (Schubert 1879)

Let C be a twisted cubic and let L_1, L_2 be two general lines. Then there are exactly 6 secant lines to C meeting both L_1 and L_2 .

Macaulay2 · symbolic coefficients

```
-- a11, a12 map to coefficients in A*(pt).
-- A is the preceding quotient ring.
use A;
F = (u,v) -> (u+v+a11*u*v
               +a12*(u^2*v+u*v^2));
nF = (n,u) -> (
  v:=0_A;
  for k from 1 to n do v=F(v,u); v);
invF = u -> (
  v:=-u;
  for k from 1 to 3 do v=v-F(u,v); v);
T = F(nF(4,alpha),invF(nF(2,e)));
T^2*alpha
```

[View output](#)
[Full script](#)

Recorded Macaulay2 output

5.

3264 and algebraic cobordism

Problem:

Can we recover the solution to Steiner's
conic problem in ?

Steiner's conic problem in algebraic cobordism

Let V be the Veronese surface of double lines.

Parameter space: the moduli space of complete conics

Over a double line ℓ , the blowup has fiber

A point over ℓ records the double line together with a normal direction.



Oriented cohomology ring of

Let X have coefficient ring R and formal group law

Set $\mathcal{K}(X)$, and $\mathcal{K}(X)$ on $\mathcal{K}(X)$. Then

Steps to applying the full blowup presentation formula:

1. Compute the **normal bundle** $N_{X/Y}$.
2. Use the **projective bundle formula** for $\mathcal{K}(N_{X/Y})$, where $\mathcal{K}(N_{X/Y})$; compute powers and the formal inverse of $\mathcal{K}(N_{X/Y})$.
3. Compute pushforwards from the blowup center X in preparation of excess intersection.
4. Take **pullbacks from** $\mathcal{K}(N_{X/Y})$ and **pushforwards from** $\mathcal{K}(X)$ as generators, and impose relations (i)–(iv).

Oriented cohomology ring of

Normal bundle of

Proposition 5.3 (Tyurin 1980)

Let ν be the quadratic Veronese embedding, and let $\mathcal{N}$ be the Veronese surface. The normal bundle of this embedding is

Its Chern classes are

Projective bundle formula:

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Oriented cohomology ring of

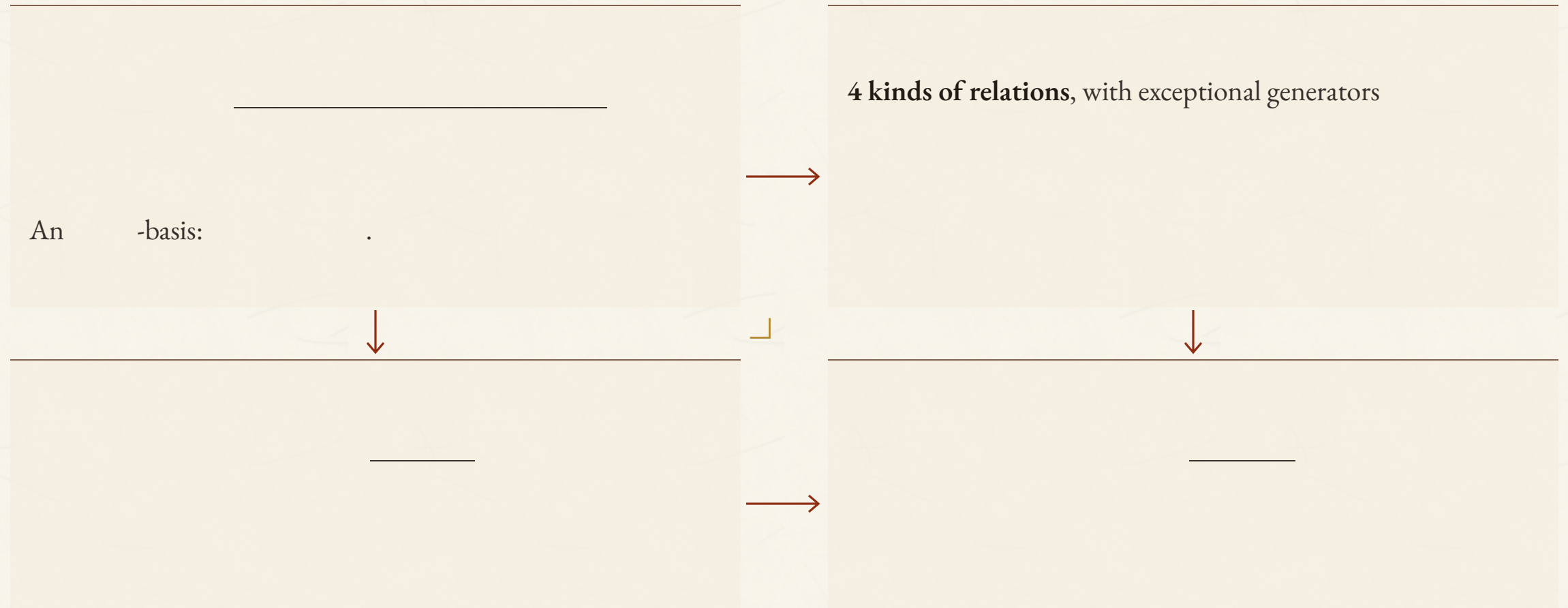
Fundamental classes from

Proposition 5.9

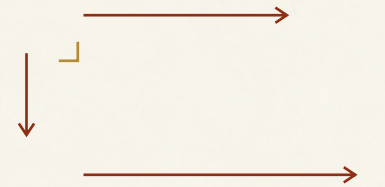
Let α and β . Then in $H^*(X, \mathbb{Z})$, one has

Oriented cohomology ring of

Here $\sigma_1, \sigma_2, \sigma_3$ are the hyperplane classes on $\mathbb{P}^1, \mathbb{P}^1, \mathbb{P}^1$, and σ_4 . We also write σ_i for their pullbacks by π_i .



Oriented cohomology ring of



(i) Intersection on

Pullback is a ring homomorphism:

The only pullback relation is therefore

Oriented cohomology ring of



(ii) Intersection between and

Projection formula, with :

For , this gives the mixed relations

Here for , since .

Oriented cohomology ring of



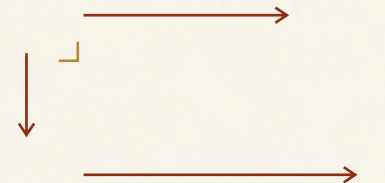
(iii) Intersection on

Self-intersection formula, for :

For , the exceptional products are

where . Higher indices mean

Oriented cohomology ring of



(iv) Excess intersection

The normal bundle has rank three, so the excess bundle has rank two:

Its top Chern class is

Family (iv) identifies ambient and exceptional classes:

Oriented cohomology ring of



(iv) Excess intersection

Evaluate on $\sigma_1, \sigma_2, \sigma_3$, the three basis classes of $H^*(\mathbb{P}^3)$. Since σ_1 represents a point of $\mathbb{P}^3$,

The remaining excess relations, after substituting $\sigma_1 = 0$, are

Thus σ_2 is eliminated, exactly as the point generator was for the twisted cubic.

Oriented cohomology ring of

Theorem 5.10 (Debnath–Z.)

With π^* and π_* defined in the blowup square, there is an isomorphism of graded $\mathbb{Z}$ -algebras

$$\left(\begin{array}{c} \text{ } \end{array} \right)$$

Here π^* ; use π_* and the higher-index classes as defined in (iii).

Steiner's conic problem in algebraic cobordism

For a general smooth conic C , let S_C be the sextic of conics tangent to C .

It has multiplicity two along C . For its strict transform $\tilde{S}_C$,
 $\tilde{S}_C \sim 6\tilde{C}$

Let $\tilde{H}$ be the pullback of a general hyperplane in $\mathbb{P}^2$. Then
 $\tilde{H} \sim \tilde{C}$

Steiner's conic problem in algebraic cobordism

For the count, take $\mathcal{C}$, with coefficients in the Lazard ring $\mathbb{L}$.

For a general conic C , the strict transform $\tilde{C}$ has

Write $\tilde{C} = C + \lambda L$ and $\tilde{C}^2 = C^2 + 2\lambda C \cdot L + \lambda^2 L^2$. The universal formal group law gives

$$\tilde{C}^2 = C^2 + 2\lambda C \cdot L + \lambda^2 L^2$$

Steiner's conic problem in algebraic cobordism

The dimension axiom kills products of six divisor classes, so all higher-order terms disappear in $\mathcal{A}^*$.

Theorem 5.12 (Chasles 1864)

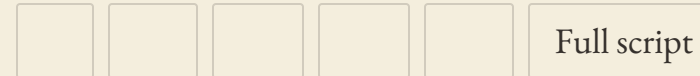
There are $2^6 = 64$ conics simultaneously tangent to 6 general conics in $\mathbb{P}^2$, which are not double lines.

Inspect $\mathcal{A}^*$: the $\mathcal{A}^6$ -terms disappear in the fifth power.

[View the Macaulay2 script and recorded outputs](#)

Macaulay2 · powers of the tangency class

```
use AB1VP5;
T = F(nF(6,alpha),invF(nF(2,e00)));
T
T^2
T^3
T^4
T^5
```



Recorded Macaulay2 output

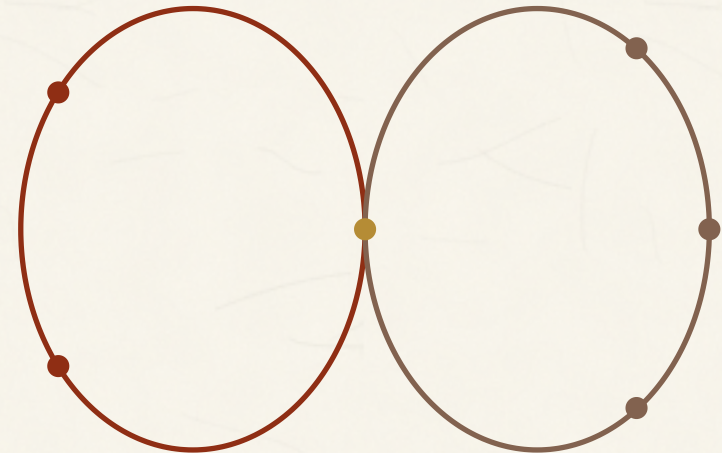
6.

Oriented cohomology rings of

Oriented cohomology ring of $\overline{\mathcal{M}}_{g,n}$

For $g \geq 0$, $\overline{\mathcal{M}}_{g,n}$ parametrizes stable n -pointed genus-zero curves. Each component has at least three marked points or nodes.

The diagram lies in the boundary divisor δ :
 markings $\bullet$ are on one component, and $\bullet$ on the other.



Oriented cohomology ring of $\overline{\mathcal{M}}_{g,n}$

Let $\mathcal{M}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}$, with $\pi: \overline{\mathcal{M}}_{g,n} \rightarrow \mathcal{M}_{g,n}$. The boundary divisor δ separates the markings from ∞ .

Each extra marking records the gluing node.

Proposition 6.4

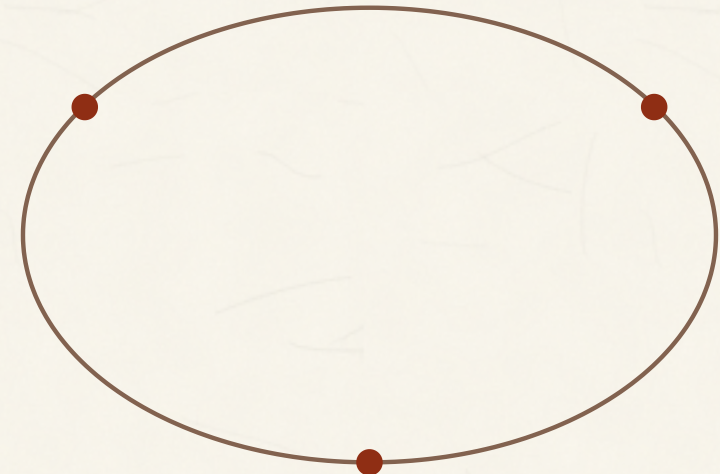
The oriented cohomology ring $h^*(\overline{\mathcal{M}}_{g,n})$ is generated as an $h^*(\text{pt})$ -algebra by the fundamental classes ψ_i of the boundary divisors.

Oriented cohomology ring of $\overline{\mathcal{M}}_{g,n}$

The forgetful morphism $\pi: \overline{\mathcal{M}}_{g,n} \rightarrow \overline{\mathcal{M}}_g$ retains four distinct markings and stabilizes the curve:

The boundary point σ_1 separates $\overline{\mathcal{M}}_{g,n}$ from $\overline{\mathcal{M}}_{g,n-1}$; its pullback is σ_1^*

The three boundary points on σ_1^* are linearly equivalent, so their pullbacks are linearly equivalent divisors.



Oriented cohomology ring of $\overline{M}_{g,n}$

$+$ denotes iterated addition by 1 . Fix distinct $i_1, \dots, i_n$.

Proposition 6.6

The equality

holds in $\overline{M}_{g,n}$.

Oriented cohomology ring of $\overline{\mathcal{M}}_{g,n}$

For stable subsets σ, τ , the same divisor has two labels, while disjoint divisors have zero product:

The subsets $\sigma, \tau, \rho, \lambda$ are **incompatible** precisely when all four sets $\sigma \cap \tau$, $\sigma \cap \rho$, $\sigma \cap \lambda$, $\tau \cap \rho$ are nonempty.

Since $\dim \overline{\mathcal{M}}_{g,n} = 3g-3+n$, the dimension axiom gives

Individual nilpotence makes every formal sum in the finitely many variables ψ, σ_i, τ_i a finite polynomial.

Oriented cohomology ring of $\overline{\mathcal{M}}_{g,n}$

Keel (1992)



Debnath–Z.



GP26: Exceptional Isomorphisms in Cobordism

Let $\mathcal{A}$ be a complex hyperplane arrangement, let $\mathcal{B}$ be its wonderful variety of the maximal building set.

Gandhi-Partida proved that there is an isomorphism of $\mathbb{Z}$ -algebras:

For $\mathcal{A}$, the wonderful model uses the minimal building set. The same argument should apply directly to $\mathcal{B}$, giving the expected isomorphism

$$\mathbb{Z}[\mathcal{A}] \cong \mathbb{Z}[\mathcal{B}]$$

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3264 and Algebraic Cobordism

Michael Ruofan Zeng

Based on joint work with Arkamouli Debnath, arXiv:2604.14536:

"Oriented cohomology rings of some moduli spaces via blowups."

University of Washington, Seattle



[Preprint · arXiv:2604.14536](https://arxiv.org/abs/2604.14536)